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www.elsevier.com/locate/jalgebraOn integrals and cointegrals for quasi-Hopf algebras [☆]Daniel Bulacu ^{a,*}, Stefaan Caenepeel ^b^a Faculty of Mathematics and Informatics, University of Bucharest, Str. Academiei 14, RO-010014 Bucharest 1, Romania^b Faculty of Engineering, Vrije Universiteit Brussel, B-1050 Brussels, Belgium

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ABSTRACT

Using the theory of Frobenius algebras, we study how the antipode of a quasi-Hopf algebra H acts on the space of left and right integrals and cointegrals. We obtain formulas that allow us to find out the explicit form of the integrals and cointegrals for the Drinfeld double $D(H)$, in terms of the integrals and cointegrals of H . This leads to an answer to a conjecture made by Hausser and Nill at the end of the nineties.

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1. Introduction

The main aim of this paper is to give an answer to the following conjecture raised by Hausser and Nill in [11]: if λ is a non-zero left cointegral on a finite dimensional quasi-Hopf algebra H and r is a non-zero right integral in H then $\beta \mapsto \lambda \bowtie r$ is a non-zero left integral in the quantum double $D(H)$ of H , at least if H is unimodular; moreover, $D(H)$ is always a unimodular quasi-Hopf algebra. Here β and α are the distinguished elements of H that appear in the definition of the antipode S of H .

It was shown in [4, Theorem 6.5] that the Drinfeld double $D(H)$ is unimodular. This goes in two steps: first it is shown that $D(H)$ is a factorizable quasi-Hopf algebra, and then it is shown that every factorizable quasi-Hopf algebra is unimodular. A categorical version of this result was given in [8],

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* Corresponding author.

E-mail addresses: daniel.bulacu@fmi.unibuc.ro (D. Bulacu), scaenepe@vub.ac.be (S. Caenepeel).

where it was shown that a factorizable braided tensor category is unimodular. Anyway, in the quasi-Hopf algebra case this braided monoidal approach was necessary because an explicit description of an integral in $D(H)$ was not available at that time. We will provide such a description: we will show that $\mu^{-1}(\delta^2)\delta^1 \rightharpoonup \lambda \bowtie r$ is a left and right integral in $D(H)$, where μ is the modular element of H^* and $\delta^1 \otimes \delta^2$ is a certain element in $H \otimes H$ introduced by Drinfeld in [6], see (2.10). When H is unimodular we have that $\mu^{-1}(\delta^2)\delta^1 = \beta$, and therefore the conjecture of Hausser and Nill is true in this case. Furthermore, using the Maschke theorem from [16] we obtain that $D(H)$ is semisimple if and only if H is semisimple and admits a normalized left cointegral, that is a left cointegral λ on H satisfying $\lambda(S^{-1}(\alpha)\beta) \neq 0$. This improves [11, Corollary 8.3]. Furthermore, we will describe explicitly the form of a left or right cointegral on $D(H)$ in terms of certain integrals and cointegrals for H , and then it will come out that $D(H)$ has a normalized left cointegral if and only if H is semisimple and H has a left normalized cointegral, if and only if $D(H)$ is a semisimple algebra. In particular, this gives a partial answer to a conjecture in [5].

A first question that we will deal with is the following: what is the appropriate notion of a right cointegral on H ? In classical Hopf algebra theory, this is a left cointegral on H^{cop} or $H^{\text{op}, \text{cop}}$, the opposite, respectively the opposite, coopposite Hopf algebra associated to H . However, if H is a quasi-Hopf algebra, then the notions of left cointegral on H^{cop} and $H^{\text{op}, \text{cop}}$ are different, and we have to decide which choice is the appropriate one. We have chosen the “cop”-version, motivated on one hand by a result of Pareigis [15] that was recently updated by Schauenburg [18], and on the other hand by explicit formulas that we obtained in Proposition 3.2. This allows easy characterizations for an element of H^* to be a left or right cointegral on H . Roughly speaking, these characterizations tell us that it suffices to check the defining condition of a cointegral on a non-zero left integral in H , rather than on all elements of H . As an application, we are able to compute the space of left cointegrals of $H(2)$ and $H_{\pm}(8)$, two quasi-Hopf algebras that have been introduced in [7].

The spaces of left and right cointegrals on H , \mathcal{L} and \mathcal{R} , are one-dimensional, and therefore isomorphic. In Section 4, we construct explicit isomorphisms, using the antipode \bar{S} of H^* or its inverse. We use techniques coming from the theory of Frobenius algebras, as developed in [12, 19]. The results can be applied to compute the space of right cointegrals on the quasi-Hopf algebras $H(2)$ and $H_{\pm}(8)$; moreover, we find some new formulas for the fourth power of the antipode of H in the case where $\mathcal{L} = \mathcal{R}$. The images of left and right integrals in H under the antipode and its inverse will be presented in Proposition 4.12. Our final result is a description of the modular elements of $D(H)$ and $D(H)^*$ in terms of the modular elements of H and H^* .

We end our introduction with a philosophical note. Although the definition of quasi-Hopf algebras is – essentially – very natural, the explicit formulas and computations are often quite technical. In order to streamline the storyline of this paper, we therefore have decided to move some of the more technical computations to Appendix A.

2. Preliminary results

We work over a commutative field k . All algebras, linear spaces, etc. will be over k ; unadorned \otimes means \otimes_k . Following Drinfeld [6], a quasi-bialgebra is a four-tuple $(H, \Delta, \varepsilon, \Phi)$ where H is an associative algebra with unit, Φ is an invertible element in $H \otimes H \otimes H$, and $\Delta: H \rightarrow H \otimes H$ and $\varepsilon: H \rightarrow k$ are algebra homomorphisms satisfying the identities

$$(\text{Id}_H \otimes \Delta)(\Delta(h)) = \Phi(\Delta \otimes \text{Id}_H)(\Delta(h))\Phi^{-1}, \quad (2.1)$$

$$(\text{Id}_H \otimes \varepsilon)(\Delta(h)) = h, \quad (\varepsilon \otimes \text{Id}_H)(\Delta(h)) = h, \quad (2.2)$$

for all $h \in H$, where Φ is a 3-cocycle, in the sense that

$$(1 \otimes \Phi)(\text{Id}_H \otimes \Delta \otimes \text{Id}_H)(\Phi)(\Phi \otimes 1) = (\text{Id}_H \otimes \text{Id}_H \otimes \Delta)(\Phi)(\Delta \otimes \text{Id}_H \otimes \text{Id}_H)(\Phi), \quad (2.3)$$

$$(\text{Id} \otimes \varepsilon \otimes \text{Id}_H)(\Phi) = 1 \otimes 1. \quad (2.4)$$

The map Δ is called the coproduct or the comultiplication, ε the counit and Φ the reassociator. As for Hopf algebras we denote $\Delta(h) = h_1 \otimes h_2$, but since Δ is only quasi-coassociative we adopt the further convention (summation understood):

$$(\Delta \otimes \text{Id}_H)(\Delta(h)) = h_{(1,1)} \otimes h_{(1,2)} \otimes h_2, \quad (\text{Id}_H \otimes \Delta)(\Delta(h)) = h_1 \otimes h_{(2,1)} \otimes h_{(2,2)},$$

for all $h \in H$. We will denote the tensor components of Φ by capital letters, and the ones of Φ^{-1} by small letters, namely

$$\begin{aligned} \Phi &= X^1 \otimes X^2 \otimes X^3 = T^1 \otimes T^2 \otimes T^3 = V^1 \otimes V^2 \otimes V^3 = \dots, \\ \Phi^{-1} &= x^1 \otimes x^2 \otimes x^3 = t^1 \otimes t^2 \otimes t^3 = v^1 \otimes v^2 \otimes v^3 = \dots. \end{aligned}$$

H is called a quasi-Hopf algebra if, moreover, there exists an anti-morphism S of the algebra H and elements $\alpha, \beta \in H$ such that, for all $h \in H$, we have

$$S(h_1)\alpha h_2 = \varepsilon(h)\alpha \quad \text{and} \quad h_1\beta S(h_2) = \varepsilon(h)\beta, \quad (2.5)$$

$$X^1\beta S(X^2)\alpha X^3 = 1 \quad \text{and} \quad S(x^1)\alpha x^2\beta S(x^3) = 1. \quad (2.6)$$

Our definition of a quasi-Hopf algebra is different from the one given by Drinfeld [6] in the sense that we do not require the antipode to be bijective. In the case where H is finite dimensional or quasi-triangular, bijectivity of the antipode follows from the other axioms, see [2] and [3], so that both definitions coincide.

For later use, we now recall some examples of quasi-Hopf algebras; they appeared for the first time in [7], and can be considered as the first explicit examples of quasi-Hopf algebras. The integral and cointegral theory that we will develop will be applied to these examples.

Example 2.1. For k a field of characteristic different from 2, let $H(2) = kC_2$, where C_2 is the cyclic group of order two generated by an element g . Since $H(2)$ is commutative it can be also viewed as a quasi-Hopf algebra with reassociator $\Phi = 1 - 2p_- \otimes p_- \otimes p_-$, antipode defined by $S(g) = g$ and distinguished elements $\alpha = g$ and $\beta = 1$. Here $p_- = \frac{1}{2}(1 - g)$.

Example 2.2. Consider k a field that contains a primitive fourth root of unit i (in particular, the characteristic of k is not 2). Let $H_{\pm}(8)$ be the unital algebra generated by g, x with relations $g^2 = 1$, $x^4 = 0$ and $gx = -xg$, and endowed with the (non-coassociative) coalgebra structure given by the formulas

$$\Delta(g) = g \otimes g, \quad \varepsilon(g) = 1, \quad \Delta(x) = x \otimes (p_+ \pm ip_-) + 1 \otimes p_+x + g \otimes p_-x, \quad \varepsilon(x) = 0,$$

where $p_{\pm} = \frac{1}{2}(1 \pm g)$. Then $H_{\pm}(8)$ are 8-dimensional quasi-Hopf algebras with reassociator $\Phi = 1 - 2p_- \otimes p_- \otimes p_-$, antipode defined by $S(g) = g$ and $S(x) = -x(p_+ \pm ip_-)$, and distinguished elements $\alpha = g$ and $\beta = 1$.

More examples of quasi-Hopf algebras can be obtained by considering the opposite or coopposite construction. More precisely, if $H = (H, \Delta, \varepsilon, \Phi, S, \alpha, \beta)$ is a quasi-bialgebra or a quasi-Hopf algebra then $H^{\text{op}}, H^{\text{cop}}$ and $H^{\text{op.cop}}$ are also quasi-bialgebras (respectively quasi-Hopf algebras), where “op” means opposite multiplication and “cop” means opposite comultiplication. The structure maps are given by $\Phi_{\text{op}} = \Phi^{-1}$, $\Phi_{\text{cop}} = (\Phi^{-1})^{321}$, $\Phi_{\text{op.cop}} = \Phi^{321}$, $S_{\text{op}} = S_{\text{cop}} = (S_{\text{op.cop}})^{-1} = S^{-1}$, $\alpha_{\text{op}} = S^{-1}(\beta)$, $\beta_{\text{op}} = S^{-1}(\alpha)$, $\alpha_{\text{cop}} = S^{-1}(\alpha)$, $\beta_{\text{cop}} = S^{-1}(\beta)$, $\alpha_{\text{op.cop}} = \beta$ and $\beta_{\text{op.cop}} = \alpha$.

The axioms of a quasi-Hopf algebra imply that $\varepsilon \circ S = \varepsilon$ and $\varepsilon(\alpha)\varepsilon(\beta) = 1$, so, by rescaling α and β , we can assume without loss of generality that $\varepsilon(\alpha) = \varepsilon(\beta) = 1$. The identities (2.2)–(2.4) also imply that

$$(\varepsilon \otimes \text{Id}_H \otimes \text{Id}_H)(\Phi) = (\text{Id}_H \otimes \text{Id}_H \otimes \varepsilon)(\Phi) = 1 \otimes 1. \quad (2.7)$$

It is well known that the antipode of a Hopf algebra is an anti-coalgebra morphism. For a quasi-Hopf algebra, we have the following statement: there exists an invertible element $f = f^1 \otimes f^2 \in H \otimes H$, called the Drinfeld twist or gauge transformation, such that $\varepsilon(f^1)f^2 = \varepsilon(f^2)f^1 = 1$ and

$$f \Delta(S(h))f^{-1} = (S \otimes S)(\Delta^{\text{cop}}(h)), \quad (2.8)$$

for all $h \in H$. f can be described explicitly: first we define $\gamma, \delta \in H \otimes H$ by

$$\gamma = S(x^1 X^2) \alpha x^2 X_1^3 \otimes S(X^1) \alpha x^3 X_2^3 \stackrel{(2.3), (2.5)}{=} S(X^2 x_2^1) \alpha X^3 x^2 \otimes S(X^1 x_1^1) \alpha x^3, \quad (2.9)$$

$$\delta = X_1^1 x^1 \beta S(X^3) \otimes X_2^1 x^2 \beta S(X^2 x^3) \stackrel{(2.3), (2.5)}{=} x^1 \beta S(x_2^3 X^3) \otimes x^2 X^1 \beta S(x_1^3 X^2). \quad (2.10)$$

With this notation f and f^{-1} are given by the formulas

$$f = (S \otimes S)(\Delta^{\text{op}}(x^1)) \gamma \Delta(x^2 \beta S(x^3)), \quad (2.11)$$

$$f^{-1} = \Delta(S(x^1) \alpha x^2) \delta (S \otimes S)(\Delta^{\text{cop}}(x^3)). \quad (2.12)$$

Moreover, f satisfies the following relations:

$$f \Delta(\alpha) = \gamma, \quad \Delta(\beta) f^{-1} = \delta, \quad (2.13)$$

$$f^1 X^1 \otimes F^1 f_1^2 X^2 \otimes F^2 f_2^2 X^3 = S(X^3) f^1 F_1^1 \otimes S(X^2) f^2 F_2^1 \otimes S(X^1) F^2, \quad (2.14)$$

$$g^1 S(g^2 \alpha) = \beta, \quad S(\beta f^1) f^2 = \alpha, \quad f^1 \beta S(f^2) = S(\alpha), \quad (2.15)$$

where we have denoted $f = f^1 \otimes f^2 = F^1 \otimes F^2$ and $f^{-1} = g^1 \otimes g^2$ as elements in $H \otimes H$. The proof of these equations can be found in [6] and [1, Lemma 2.6].

We will need the appropriate generalization of the formula $h_1 \otimes h_2 S(h_3) = h \otimes 1$ in classical Hopf algebra theory.

Following [9,10], we define

$$p_R = p^1 \otimes p^2 = x^1 \otimes x^2 \beta S(x^3), \quad (2.16)$$

$$q_R = q^1 \otimes q^2 = X^1 \otimes S^{-1}(\alpha X^3) X^2, \quad (2.17)$$

$$p_L = \tilde{p}^1 \otimes \tilde{p}^2 = X^2 S^{-1}(X^1 \beta) \otimes X^3, \quad (2.18)$$

$$q_L = \tilde{q}^1 \otimes \tilde{q}^2 = S(x^1) \alpha x^2 \otimes x^3. \quad (2.19)$$

For all $h \in H$, we then have

$$\Delta(h_1) p_R(1 \otimes S(h_2)) = p_R(h \otimes 1), \quad (2.20)$$

$$(1 \otimes S^{-1}(h_2)) q_R \Delta(h_1) = (h \otimes 1) q_R, \quad (2.21)$$

$$\Delta(h_2) p_L(S^{-1}(h_1) \otimes 1) = p_L(1 \otimes h), \quad (2.22)$$

$$(S(h_1) \otimes 1) q_L \Delta(h_2) = (1 \otimes h) q_L. \quad (2.23)$$

Furthermore, the following relations hold

$$(1 \otimes S^{-1}(p^2))q_R \Delta(p^1) = 1 \otimes 1, \quad (2.24)$$

$$\Delta(q^1)p_R(1 \otimes S(q^2)) = 1 \otimes 1, \quad (2.25)$$

$$(S(\tilde{p}^1) \otimes 1)q_L \Delta(\tilde{p}^2) = 1 \otimes 1, \quad (2.26)$$

$$\Delta(\tilde{q}^2)p_L(S^{-1}(\tilde{q}^1) \otimes 1) = 1 \otimes 1. \quad (2.27)$$

We also have that

$$X^1 p_1^1 p^1 \otimes X^2 p_2^1 p^2 \otimes X^3 p^2 = x_1^1 p^1 \otimes x_{(2,1)}^1 p_1^2 g^1 S(x^3) \otimes x_{(2,2)}^1 p_2^2 g^2 S(x^2), \quad (2.28)$$

$$q^1 Q_1^1 x^1 \otimes q^2 Q_2^1 x^2 \otimes Q^2 x^3 = q^1 X_1^1 \otimes S^{-1}(f^2 X^3) q_1^2 X_{(2,1)}^1 \otimes S^{-1}(f^1 X^2) q_2^2 X_{(2,2)}^1, \quad (2.29)$$

$$x^1 \tilde{p}^1 \otimes x^2 \tilde{p}_1^2 \tilde{p}^1 \otimes x^3 \tilde{p}_2^2 \tilde{p}^2 = X_{(1,1)}^3 \tilde{p}_1^1 S^{-1}(X^2 g^2) \otimes X_{(1,2)}^3 \tilde{p}_2^1 S^{-1}(X^1 g^1) \otimes X_2^3 \tilde{p}^2, \quad (2.30)$$

$$\tilde{Q}^1 X^1 \otimes \tilde{q}^1 \tilde{Q}_1^2 X^2 \otimes \tilde{q}^2 \tilde{Q}_2^2 X^3 = S(x^2) f^1 \tilde{q}_1^1 x_{(1,1)}^3 \otimes S(x^1) f^2 \tilde{q}_2^1 x_{(1,2)}^3 \otimes \tilde{q}^2 x_2^3. \quad (2.31)$$

3. Left and right cointegrals on a quasi-Hopf algebra

Hausser and Nill [11] have introduced the notion of left integral on a finite dimensional quasi-Hopf algebra H . They define the space of left cointegrals \mathcal{L} on H as the set of certain coinvariants associated to the quasi-Hopf H -bimodule H^* , the linear dual of H . Schauenburg [18] has observed that the affiliation of H^* to the category of right quasi-Hopf H -bimodules is a consequence of the following monoidal categorical result due to Pareigis [15]. For details on rigid monoidal categories, we refer the reader to [13,14].

Proposition 3.1. *Let C be a coalgebra in the monoidal category $\mathcal{C} = (\mathcal{C}, \otimes, a, \underline{1}, l, r)$.*

- (i) *Let (V, λ_V) be a left C -comodule admitting a right dual *V , with evaluation and coevaluation morphisms ev'_V and coev'_V . Then *V is a right C -comodule, with right C -coaction*

$$\begin{aligned} {}^*V &\xrightarrow{l_V^{-1}} \underline{1} \otimes {}^*V \xrightarrow{\text{coev}'_V \otimes \text{Id}_{{}^*V}} ({}^*V \otimes V) \otimes {}^*V \xrightarrow{(\text{Id}_{{}^*V} \otimes \lambda_V) \otimes \text{Id}_{{}^*V}} ({}^*V \otimes (C \otimes V)) \otimes {}^*V \\ &\xrightarrow{a_{V,V,C}^{-1} \otimes \text{Id}_{{}^*V}} (({}^*V \otimes C) \otimes V) \otimes {}^*V \xrightarrow{a_{V \otimes C, V, {}^*V}} ({}^*V \otimes C) \otimes (V \otimes {}^*V) \\ &\xrightarrow{\text{Id}_{{}^*V \otimes C} \otimes \text{ev}'_V} ({}^*V \otimes C) \otimes \underline{1} \xrightarrow{r_{V \otimes C}} {}^*V \otimes C. \end{aligned}$$

- (ii) *In a similar way, if (V, ρ_V) is a right C -comodule admitting a left dual V^* , with evaluation and coevaluation morphisms ev_V and coev_V , then V^* is a left C -comodule, with left C -coaction*

$$\begin{aligned} V^* &\xrightarrow{r_{V^*}^{-1}} V^* \otimes \underline{1} \xrightarrow{\text{Id}_{V^*} \otimes \text{coev}_V} V^* \otimes (V \otimes V^*) \xrightarrow{\text{Id}_{V^*} \otimes (\rho_V \otimes \text{Id}_{V^*})} V^* \otimes ((V \otimes C) \otimes V^*) \\ &\xrightarrow{\text{Id}_{V^*} \otimes a_{V,C,V^*}} V^* \otimes (V \otimes (C \otimes V^*)) \xrightarrow{a_{V^*,V,C \otimes V^*}^{-1}} (V^* \otimes V) \otimes (C \otimes V^*) \\ &\xrightarrow{\text{ev}_V \otimes \text{Id}_{C \otimes V^*}} \underline{1} \otimes (C \otimes V^*) \xrightarrow{l_{C \otimes V^*}} C \otimes V^*. \end{aligned}$$

We apply this general result for the case when $\mathcal{C} = {}_H\mathcal{M}_H$, the category of H -bimodules. This category is monoidal since it can be identified with the category of left modules over the quasi-Hopf algebra $H^{\text{op}} \otimes H$. We provide the explicit construction of the monoidal structure on ${}_H\mathcal{M}_H$.

- The associativity constraints $a_{M,N,P} : (M \otimes N) \otimes P \rightarrow M \otimes (N \otimes P)$ are given by

$$a_{M,N,P}((m \otimes n) \otimes p) = X^1 \cdot m \cdot x^1 \otimes (X^2 \cdot n \cdot x^2 \otimes X^3 \cdot p \cdot x^3); \quad (3.1)$$

- the unit object is k viewed as an H -bimodule via the counit ε of H ;
- the left and right unit constraints are given by the natural isomorphisms $k \otimes M \cong M \cong M \otimes k$.

Recall that the linear dual V^* of a right (respectively left) H -module V is a left (respectively right) H^* -module via $(h \rightharpoonup v^*)(v) = v^*(v \cdot h)$ (respectively $(v^* \leftharpoonup h)(v) = v^*(h \cdot v)$).

Let $\{v_i\}_i$ be a basis of a finite dimensional H -bimodule V , with dual basis $\{v^i\}_i$ of its linear dual V^* . The left dual of V is V^* , with H -bimodule structure

$$h \cdot v^* \cdot h' = (h' \otimes h) \cdot v^* = v^* \leftharpoonup (S^{-1}(h') \otimes S(h)) = S^{-1}(h') \rightharpoonup v^* \leftharpoonup S(h). \quad (3.2)$$

The evaluation morphism $\text{ev}_V : V^* \otimes V \rightarrow k$ and the coevaluation morphism $\text{coev}_V : k \rightarrow V \otimes V^*$ are given by the formulas

$$\text{ev}_V(v^* \otimes v) = v^*((S^{-1}(\beta) \otimes \alpha) \cdot v) = v^*(\alpha \cdot v \cdot S^{-1}(\beta)); \quad (3.3)$$

$$\text{coev}_V(1) = \sum_i (S^{-1}(\alpha) \otimes \beta) \cdot v_i \otimes v^i = \sum_i \beta \cdot v_i \cdot S^{-1}(\alpha) \otimes v^i. \quad (3.4)$$

The right dual of V is V^* , now with H -bimodule structure

$$h \cdot v^* \cdot h' = (h' \otimes h) \cdot v^* = v^* \leftharpoonup (S(h') \otimes S^{-1}(h)) = S(h') \rightharpoonup v^* \leftharpoonup S^{-1}(h).$$

The evaluation $\text{ev}'_V : V \otimes V^* \rightarrow k$ and coevaluation $\text{coev}'_V : k \rightarrow V^* \otimes V$ are now given by

$$\text{ev}'_V(v^* \otimes v) = v^*((\beta \otimes S^{-1}(\alpha)) \cdot v) = v^*(S^{-1}(\alpha) \cdot v \cdot \beta);$$

$$\text{coev}'_V(1) = \sum_i v^i \otimes (\alpha \otimes S^{-1}(\beta)) \cdot v_i = \sum_i v^i \otimes S^{-1}(\beta) \cdot v_i \cdot \alpha.$$

Note that the two assertions follow easily from the canonical monoidal identification ${}_H\mathcal{M}_H = {}^{H^{\text{op}}}\otimes {}_H\mathcal{M}$ and the rigid monoidal structure of the category of finite dimensional representations over a quasi-Hopf algebra. The verification of all these details is left to the reader.

Via its (quasi-)coalgebra structure H has a natural (monoidal) coalgebra structure within ${}_H\mathcal{M}_H$. Recall that the category of right (left) quasi-Hopf H -bimodules is precisely the category of right (left) H -comodules within ${}_H\mathcal{M}_H$. The explicit definition of these concepts can be found in [11]. Note that a left quasi-Hopf H -bimodule is nothing else than a right quasi-Hopf H^{cop} -bimodule.

Proposition 3.2. *Let H be a finite dimensional quasi-Hopf algebra, with basis $\{e_i\}_i$, and let $\{e^i\}_i$ be the corresponding dual basis of H^* .*

- (i) H^* is a left quasi-Hopf H -bimodule via the structure

$$h \cdot h^* \cdot h' = S^{-1}(h') \rightharpoonup h^* \leftharpoonup S(h);$$

$$\lambda_{H^*}(h^*) = \sum_i h^*(S(\tilde{p}^2)f^1(e_i)_1S^{-1}(\tilde{q}^2g^2))S(\tilde{p}^1)f^2(e_i)_2S^{-1}(\tilde{q}^1g^1) \otimes e^i.$$

Here $p_L = \tilde{p}^1 \otimes \tilde{p}^2$ and $q_L = \tilde{q}^1 \otimes \tilde{q}^2$ are the elements defined in (2.18)–(2.19), $f = f^1 \otimes f^2$ is the Drinfeld twist from (2.11) and $f^{-1} = g^1 \otimes g^2$ is its inverse from (2.12).

(ii) H^* is a right quasi-Hopf H -bimodule via the structure

$$h \cdot h^* \cdot h' = S(h') \rightharpoonup h^* \leftharpoonup S^{-1}(h),$$

$$\rho_{H^*}(h^*) = \sum_i h^*(S^{-1}(f^1 p^1)(e_i)_2 g^2 S(q^1)) e^i \otimes S^{-1}(f^2 p^2)(e_i)_1 g^1 S(q^2),$$

where $p_R = p^1 \otimes p^2$ and $q_R = q^1 \otimes q^2$ are the elements presented in (2.16)–(2.17).

Proof. We will prove (i), and leave (ii) to the reader. Actually, we will show that the structure on H^* as stated in (i) is precisely the structure that we obtain after applying part (ii) of Proposition 3.1 in the case where $\mathcal{C} = {}_H\mathcal{M}_H$ and $V = H$. Consider H as a bimodule via multiplication. As we have already mentioned the coassociativity of Δ expresses the fact that H is a coalgebra within ${}_H\mathcal{M}_H$, and so a right H -comodule in \mathcal{C} . Since H is finite dimensional we get that H^* , the left dual of H , is a left H -comodule in \mathcal{C} . By (3.2) we deduce that the H -bimodule structure of H^* is the one mentioned in part (i) of the statement. In order to find the left H -coaction on H^* we specialize Proposition 3.1 (ii) for the monoidal structure of ${}_H\mathcal{M}_H$, and use (3.3) and (3.4) to compute λ_{H^*} as the following composition.

$$\begin{aligned} h^* &\xrightarrow{\text{Id}_{H^*} \otimes \text{coev}_H} \sum_i h^* \otimes (\beta e_i S^{-1}(\alpha) \otimes e^i) = \sum_{i,j} e^j (\beta e_i S^{-1}(\alpha)) h^* \otimes (e_j \otimes e^i) \\ &\xrightarrow{\text{Id}_{H^*} \otimes (\Delta \otimes \text{Id}_H)} \sum_{i,j} e^j (\beta e_i S^{-1}(\alpha)) h^* \otimes ((e_j)_1 \otimes (e_j)_2) \otimes e^i \\ &\xrightarrow{\text{Id}_{H^*} \otimes a_{H,H,H^*}} \sum_{i,j} e^j (\beta S(X^3) e_i S^{-1}(\alpha X^3)) h^* \otimes (X^1(e_j)_1 X^1 \otimes (X^2(e_j)_2 X^2 \otimes e^i)) \\ &\xrightarrow{a_{H^*,H,H \otimes H^*}^{-1}} \sum_{i,j} e^j (\beta S(y_2^3 X^3) e_i S^{-1}(\alpha X^3 Y_2^3)) (S^{-1}(Y^1) \rightharpoonup h^* \leftharpoonup S(y^1) \otimes y^2 X^1(e_j)_1 X^1 Y^2) \\ &\quad \otimes (y_1^3 X^2(e_j)_2 X^2 Y_1^3 \otimes e^i) \\ &\xrightarrow{\text{ev}_H \otimes \text{Id}_{H \otimes H^*}} \sum_j h^* (S(y^1) \alpha y^2 X^1(e_j)_1 X^1 Y^2 S^{-1}(Y^1 \beta)) \\ &\quad \times y_1^3 X^2(e_j)_2 X^2 Y_1^3 \otimes S^{-1}(\alpha X^3 Y_2^3) \rightharpoonup e^j \leftharpoonup \beta S(y_2^3 X^3). \end{aligned}$$

We then have, for all $h^* \in H^*$, that

$$\begin{aligned} \lambda_{H^*}(h^*) &\stackrel{(2.19)}{=} \sum_j \langle h^*, \tilde{q}^1 X^1(e_j)_1 X^1 \tilde{p}^1 \rangle \tilde{q}_1^2 X^2(e_j)_2 X^2 \tilde{p}_1^2 \otimes S^{-1}(\alpha X^3 \tilde{p}_2^2) \rightharpoonup e^j \leftharpoonup \beta S(\tilde{q}_2^2 X^3) \\ &= \sum_i \langle h^*, \tilde{q}^1 X^1 \beta_1 S(\tilde{q}_2^2 X^3)_{\cdot 1} (e_i)_1 S^{-1}(\alpha X^3 \tilde{p}_2^2)_{\cdot 1} X^1 \tilde{p}^1 \rangle \\ &\quad \times \tilde{q}_1^2 X^2 \beta_2 S(\tilde{q}_2^2 X^3)_{\cdot 2} (e_i)_2 S^{-1}(\alpha X^3 \tilde{p}_2^2)_{\cdot 2} X^2 \tilde{p}_1^2 \otimes e^i \\ &\stackrel{(2.13), (2.8)}{=} \langle h^*, \tilde{q}^1 X^1 \delta^1 S(\tilde{q}_{(2,2)}^2 X_2^3) f^1(e_i)_1 S^{-1}(\gamma^2 x_2^3 \tilde{p}_{(2,2)}^2) x^1 \tilde{p}^1 \rangle \\ &\quad \times \tilde{q}_1^2 X^2 \delta^2 S(\tilde{q}_{(2,1)}^3 X_1^3) f^2(e_i)_2 S^{-1}(\gamma^1 x_1^3 \tilde{p}_{(2,1)}^2) x^2 \tilde{p}_1^2 \otimes e^i \end{aligned}$$

$$\begin{aligned}
& \stackrel{(2.9), (2.10)}{=} \sum_i \langle h^*, \tilde{q}^1 \beta S(\tilde{q}_{(2,2)}^2 X^3) f^1(e_i)_1 S^{-1}(\alpha x^3 \tilde{p}_{(2,2)}^2 g^2) \tilde{p}^1 \rangle \\
& \quad \times \tilde{q}_1^2 X^1 \beta S(\tilde{q}_{(2,1)}^2 X^2) f^2(e_i)_2 S^{-1}(\alpha x^2 \tilde{p}_{(2,1)}^2 g^1) x^1 \tilde{p}_1^2 \otimes e^i \\
& \stackrel{(2.1), (2.5)}{=} \sum_i \langle h^*, \tilde{q}^1 \beta S(X^3 \tilde{q}^2) f^1(e_i)_1 S^{-1}(\alpha \tilde{p}^2 x^3 g^2) \tilde{p}^1 \rangle \\
& \quad \times X^1 \beta S(X^2) f^2(e_i)_2 S^{-1}(\alpha x^2 g^1) x^1 \otimes e^i \\
& \stackrel{(2.18), (2.19)}{=} \sum_i \langle h^*, S(X^3) f^1(e_i)_1 S^{-1}(x^3 g^2) \rangle X^1 \beta S(X^2) f^2(e_i)_2 S^{-1}(\alpha x^2 g^1) x^1 \otimes e^i \\
& \stackrel{(2.6)}{=} \sum_i \langle h^*, S(\tilde{p}^2) f^1(e_i)_1 S^{-1}(\tilde{q}^2 g^2) \rangle S(\tilde{p}^1) f^2(e_i)_2 S^{-1}(\tilde{q}^1 g^1) \otimes e^i,
\end{aligned}$$

as claimed. \square

If we denote

$$U = g^1 S(q^2) \otimes g^2 S(q^1) \quad \text{and} \quad V = S^{-1}(f^2 p^2) \otimes S^{-1}(f^1 p^1), \quad (3.5)$$

then the right H -coaction on H^* can be restated as

$$\rho_{H^*}(h^*) = \sum_i h^*(V^2(e_i)_2 U^2) e^i \otimes V^1(e_i)_1 U^1 = \sum_i e^i * h^* \otimes e_i,$$

where $U = U^1 \otimes U^2$, $V = V^1 \otimes V^2$ and $*$ is the – possibly non-associative – multiplication on H^* given by $(\varphi * \psi)(h) = \varphi(V^1 h_1 U^1) \psi(V^2 h_2 U^2)$. This brings us to the right quasi-Hopf H -bimodule structure on H^* introduced in [11]. The coinvariants under this coaction are called left cointegrals on H . $\lambda \in H^*$ is coinvariant, i.e. a left cointegral, if and only if

$$\lambda(V^2 h_2 U^2) V^1 h_1 U^1 = \mu(x^1) \lambda(h S(x^2)) x^3, \quad (3.6)$$

for all $h \in H$, or, according to [2, Proposition 3.4 (c)]

$$\lambda(S^{-1}(f^1) h_2 U^2) S^{-1}(f^2) h_1 U^1 = \mu(q_1^1 x^1) \lambda(h S(q_2^1 x^2)) q^2 x^3, \quad (3.7)$$

for all $h \in H$. Here μ is the so-called modular element of H^* , which can be introduced as follows. $t \in H$ is called a left (respectively right) integral in H if $ht = \varepsilon(h)t$ (respectively $th = \varepsilon(h)t$), for all $h \in H$. \int_l^H and \int_r^H , the spaces of left and right integrals are one-dimensional if H is a finite dimensional quasi-Hopf algebra, see [2,11]. They are also ideals of H , so there exists $\mu \in H^*$ such that

$$th = \mu(h)t, \quad (3.8)$$

for all $t \in \int_l^H$ and $h \in H$. It can be easily checked that μ is an algebra map and that μ is convolution invertible with inverse $\mu^{-1} = \mu \circ S = \mu \circ S^{-1}$, and that

$$hr = \mu^{-1}(h)r, \quad (3.9)$$

for all $r \in \int_r^H$ and $h \in H$. Furthermore, we have that

$$\mu(\alpha\beta)\mu^{-1}(\alpha\beta) = \mu(X^1 \beta S(X^2) \alpha X^3) \mu^{-1}(S(x^1) \alpha x^2 \beta S(x^3)) \stackrel{(2.6)}{=} 1. \quad (3.10)$$

If there exists a non-zero left integral in H which is at the same time a right integral, then H is called unimodular. Remark that H is unimodular if and only if $\mu = \varepsilon$.

Example 3.3. Consider the 2-dimensional quasi-Hopf algebra $H(2)$ constructed in Example 2.1. It can be easily checked that $t = 1 + g$ is both a non-zero left and right integral in $H(2)$. Thus $H(2)$ is a unimodular quasi-Hopf algebra, and $\mu = \varepsilon$.

Example 3.4. Let $H_{\pm}(8)$ be the two 8-dimensional quasi-Hopf algebra considered in Example 2.2. If $t = (1 + g)x^3$ then

$$\begin{aligned} gt &= g(1 + g)x^3 = (g + 1)x^3 = t = \varepsilon(g)t, \quad \text{and} \\ xt &= x(1 + g)x^3 = (x + gx)x^3 = (x - gx)x^3 = (1 - g)x^4 = 0 = \varepsilon(x)t, \end{aligned}$$

and so t is a non-zero left integral in $H_{\pm}(8)$, because g and x generate $H_{\pm}(8)$ as an algebra. In a similar way it can be shown that $r = (1 - g)x^3$ is a non-zero right integral in $H_{\pm}(8)$. Since the characteristic of k is different from 2 it follows that $H_{\pm}(8)$ is not unimodular. It can be easily seen that the modular element of $H_{\pm}(8)^*$ is given by the relations $\mu(1) = 1$, $\mu(g) = -1$ and $\mu(x) = 0$.

The space of left cointegrals on a finite dimensional quasi-Hopf algebra H is denoted by \mathcal{L} . It is proved in [11] that $\mathcal{L} \otimes H$ and H^* are isomorphic as right quasi-Hopf H -bimodules; the functional corresponding to $\lambda \otimes h \in \mathcal{L} \otimes H$ is $\lambda \cdot h = S(h) \rightarrow \lambda$. From this isomorphism, it follows easily that \mathcal{L} is one-dimensional.

Now we introduce right cointegrals. We consider the quasi-Hopf algebra H^{cop} , and observe that $\gamma_{\text{cop}} = (S^{-1} \otimes S^{-1})(\gamma)$, and so $f_{\text{cop}} = (S^{-1} \otimes S^{-1})(f)$. It is easily seen that $(p_R)_{\text{cop}} = \tilde{p}^2 \otimes \tilde{p}^1 := \tilde{p}_{21}$ and $(q_R)_{\text{cop}} = \tilde{q}^2 \otimes \tilde{q}^1 := \tilde{q}_{21}$. Otherwise stated, the left quasi-Hopf H -bimodule structure of H^* can be obtained from the right one by replacing H by H^{cop} . This is why we propose the following.

Definition 3.5. A right cointegral on a finite dimensional quasi-Hopf algebra H is a left cointegral on H^{cop} .

The space of right cointegrals on H will be denoted by \mathcal{R} . Since H^{cop} is also a finite dimensional quasi-Hopf algebra all the above results for left cointegrals can be restated for right cointegral. For example $\Lambda \in H^*$ is a right cointegral in H^* if and only if

$$\Lambda(S(\tilde{p}^2)f^1h_1S^{-1}(\tilde{q}^2g^2))S(\tilde{p}^1)f^2h_2S^{-1}(\tilde{q}^1g^1) = \mu(X^3)\Lambda(hS^{-1}(X^2))X^1,$$

for all $h \in H$. We also have that $H \otimes \mathcal{R}$ and H^* are isomorphic as left quasi-Hopf H -bimodules, where now $h \otimes \Lambda$ corresponds to $S^{-1}(h) \rightarrow \Lambda$. Consequently any non-zero right cointegral is non-degenerate and $\dim_k \mathcal{R} = 1$.

With an eye to examples, we now provide new equivalent characterizations of left cointegrals. Perhaps, the easiest characterization of a left integral λ on a Hopf algebra H is that $\lambda(t_2)t_1 = \lambda(t)1$, where t is a non-zero left integral in H . Here one of the key arguments is that t generates H as a left or right H^* -module. If H is a quasi-Hopf algebra, then H^* is not associative, and we cannot consider H^* -modules. However, we can still say that a left non-zero integral t “generates” H , in the sense that the map

$$\xi : H^* \rightarrow H, \quad \xi(h^*) = (h^* \circ S) \rightarrow t := h^*(S(q^2t_2p^2))q^1t_1p^1 \quad (3.11)$$

is bijective, cf. [2]. Moreover, ξ is a left H -module isomorphism between H and H^* .

From [11], we recall the relations

$$U[1 \otimes S(h)] = \Delta(S(h_1))U[h_2 \otimes 1], \quad (3.12)$$

$$[1 \otimes S^{-1}(h)]V = [h_2 \otimes 1]V\Delta(S^{-1}(h_1)), \quad (3.13)$$

$$q_R = [\tilde{q}^2 \otimes 1]V\Delta(S^{-1}(\tilde{q}^1)), \quad \text{and} \quad (3.14)$$

$$p_R = \Delta(S(\tilde{p}^1))U[\tilde{p}^2 \otimes 1]. \quad (3.15)$$

We will also need the following result.

Lemma 3.6. *If H is a finite dimensional quasi-Hopf algebra and μ is the modular element of H^* then we have for all $\lambda \in \mathcal{L}$ and $t \in \int_l^H$ that*

$$\lambda(q^2 t_2 p^2) q^1 t_1 p^1 = \mu(\beta) \lambda(t) 1. \quad (3.16)$$

Proof. Let us start by noting that, for all $h, h' \in H$,

$$\lambda(q^2 h_2 p^2 S(h')) q^1 h_1 p^1 = \mu(x^1) \lambda(S^{-1}(\tilde{q}^1) h S(x^2 h'_1 \tilde{p}^1)) \tilde{q}^2 x^3 h'_2 \tilde{p}^2. \quad (3.17)$$

Indeed, we have

$$\begin{aligned} \lambda(q^2 h_2 p^2 S(h')) q^1 h_1 p^1 &\stackrel{(3.14), (3.15)}{=} \lambda(V^2[S^{-1}(\tilde{q}^1) h S(\tilde{p}^1)]_2 U^2 S(h')) \tilde{q}^2 V^1[S^{-1}(\tilde{q}^1) h S(\tilde{p}^1)]_1 U^1 \tilde{p}^2 \\ &\stackrel{(3.12)}{=} \lambda(V^2[S^{-1}(\tilde{q}^1) h S(h'_1 \tilde{p}^1)]_2 U^2) \tilde{q}^2 V^1[S^{-1}(\tilde{q}^1) h S(h'_1 \tilde{p}^1)]_1 U^1 h'_2 \tilde{p}^2 \\ &\stackrel{(3.6)}{=} \mu(x^1) \lambda(S^{-1}(\tilde{q}^1) h S(x^2 h'_1 \tilde{p}^1)) \tilde{q}^2 x^3 h'_2 \tilde{p}^2. \end{aligned}$$

Specializing (3.17) for $h = t$, a left integral in H , and $h' = 1$ we obtain

$$\begin{aligned} \lambda(q^2 t_2 p^2) q^1 t_1 p^1 &\stackrel{(2.18)}{=} \mu(x^1) \lambda(t S(x^2 \tilde{p}^1)) x^3 \tilde{p}^2 \\ &\stackrel{(3.8), (2.18)}{=} \mu(x^1) \mu(X^1 \beta S(X^2)) \mu(S(x^2)) \lambda(t) x^3 X^3 = \mu(\beta) \lambda(t) 1, \end{aligned}$$

and this finishes the proof. \square

Theorem 3.7. *For a finite dimensional quasi-Hopf algebra H and a non-zero element $\lambda \in H^*$ the following assertions are equivalent:*

- (i) λ is a left cointegral on H ;
- (ii) $\lambda(q^2 t_2 p^2) q^1 t_1 p^1 = \mu(\beta) \lambda(t) 1$, for any left integral t in H ;
- (iii) $\lambda(t_2 p^2) t_1 p^1 = \mu(\beta) \lambda(t) \beta$, for all $t \in \int_l^H$;
- (iv) $\lambda(ht_2 p^2) t_1 p^1 = \mu(\beta) \lambda(t) \beta S(h)$, for all $t \in \int_l^H$ and $h \in H$.

Here p_R, q_R, U and V are the elements defined in (2.16), (2.17) and (3.5).

Proof. (i) \Rightarrow (ii) follows from (3.16).

(ii) \Rightarrow (i). Let t be a non-zero left integral in H . If $\lambda(t) = 0$ then $\lambda(Ht) = 0$, and so $\text{Ker}(\lambda)$ contains a non-zero ideal, contradicting the fact that any non-zero left cointegral on H is non-degenerate. Thus $\lambda(t) \neq 0$, and from here we obtain that $(\mu(\beta) \lambda(t))^{-1} \lambda(q^2 t_2 p^2) q^1 t_1 p^1 = 1$. Hence, by [4, Lemma 6.2] we conclude that $(\mu(\beta) \lambda(t))^{-1} \lambda$, and therefore λ itself, is a non-zero left cointegral on H .

(ii) \Rightarrow (iii). The formula

$$t_1 \otimes S(t_2) = q^1 t_1 \otimes S(q^2 t_2) \beta = \beta q^1 t_1 \otimes S(q^2 t_2) \quad (3.18)$$

can be deduced easily from (2.24) or can be found in [2, Lemma 2.1]. It implies that

$$\lambda(t_2 p^2) t_1 p^1 = \lambda(q^2 t_2 p^2) \beta q^1 t_1 p^1 \stackrel{(ii)}{=} \mu(\beta) \lambda(t) \beta.$$

(iii) \Rightarrow (iv). By (2.20) and (3.8) it follows that

$$t_1 p^1 h \otimes t_2 p^2 = \mu(h_1) t_1 p^1 \otimes t_2 p^2 S(h_2), \quad (3.19)$$

for all $h \in H$. By [2, Lemma 3.3] we also have

$$\lambda(S^{-1}(h)h') = \mu(h_1) \lambda(h' S(h_2)), \quad (3.20)$$

for all $\lambda \in \mathcal{L}$ and $h, h' \in H$. We now have, for all $h \in H$,

$$\begin{aligned} \lambda(ht_2 p^2) t_1 p^1 &= \langle \lambda, S^{-1}(S(h)) t_2 p^2 \rangle t_1 p^1 \stackrel{(3.20)}{=} \mu(S(h)_1) \langle \lambda, t_2 p^2 S(h_2) \rangle t_1 p^1 \\ &\stackrel{(3.19)}{=} \lambda(t_2 p^2) t_1 p^1 S(h) \stackrel{(iii)}{=} \mu(\beta) \lambda(t) \beta S(h). \end{aligned}$$

(iv) \Rightarrow (ii). We use (iv) to see that

$$\lambda(q^2 t_2 p^2) q^1 t_1 p^1 = \mu(\beta) \lambda(t) q^1 \beta S(q^2) \stackrel{(2.17)}{=} \mu(\beta) \lambda(t) X^1 \beta S(X^2) \alpha X^3 \stackrel{(2.6)}{=} \mu(\beta) \lambda(t) 1,$$

completing the proof. \square

The characterizations in Theorem 3.7 allow to find the left cointegrals on $H(2)$ and $H_{\pm}(8)$. First we have to find a non-zero left integral in H . Secondly, working eventually with dual bases, we have to determine the element $\lambda \in H^*$ that satisfies, for instance, (iii), the most simple equivalent condition in Theorem 3.7. When H is unimodular and α is invertible, then (iii) simplifies to $\lambda(t_2) t_1 = \lambda(t) \beta \alpha$, because $\mu = \varepsilon$, and t , which this time is also a right integral, satisfies

$$t_1 \otimes t_2 \stackrel{(2.25)}{=} t_1 q_1^1 p^1 \otimes t_2 q_2^1 p^2 S(q^2) = t_1 p^1 \otimes t_2 p^2 S(\alpha) \stackrel{(3.19)}{=} t_1 p^1 \alpha \otimes t_2 p^2.$$

Example 3.8. Let $\{P_1, P_g\}$ be the dual basis of $H(2)^*$ corresponding to the basis $\{1, g\}$ of the quasi-Hopf algebra $H(2)$ from Example 2.1. P_g is a (non-zero) left cointegral on $H(2)$.

Proof. $H(2)$ is unimodular, $\beta = 1$, $\alpha = g$ is invertible, and $t = 1 + g$ is a left and right integral, see Example 3.3. So we have to find the elements $\lambda \in H(2)^*$ satisfying $\lambda(1)1 + \lambda(g)g = \lambda(1 + g)g$. These clearly satisfy $\lambda(1) = 0$, and so $\mathcal{L} = kP_g$. \square

Example 3.9. Let $\{P_{g^i x^j} \mid 0 \leq i \leq 1, 0 \leq j \leq 3\}$ be the dual basis corresponding to the canonical basis $\{g^i x^j \mid 0 \leq i \leq 1, 0 \leq j \leq 3\}$ of $H_{\pm}(8)$. The space of left cointegrals on the quasi-Hopf algebra $H_{\pm}(8)$ from Example 2.2 is kP_{x^3} .

Proof. This time the computations are more complicated. Recall first that $t = (1 + g)x^3 = x^3(1 - g)$ is a non-zero left integral in $H_{\pm}(8)$ which is not a right integral, see Example 3.4.

Denote $\omega := \frac{1}{2}(1 \pm i)$ and let $\bar{\omega} = \frac{1}{2}(1 \mp i)$ be its conjugate. In order to compute $\Delta(t)$ we rewrite $\Delta(x)$ as

$$\Delta(x) = \omega x \otimes 1 + \bar{\omega} x \otimes g + p_+ \otimes x + p_- \otimes gx,$$

to compute that

$$\Delta(x^2) = x^2 \otimes g + g \otimes x^2 + (p_+ \pm ip_-)x \otimes x + (p_- \pm ip_+)x \otimes gx,$$

and then that

$$\begin{aligned} \Delta(x^3) &= \bar{\omega} x^3 \otimes 1 + \omega x^3 \otimes g \pm ip_- x^2 \otimes x + \bar{\omega} gx \otimes x^2 + p_+ \otimes x^3 \\ &\quad \pm ip_+ x^2 \otimes gx - \omega gx \otimes gx^2 - p_- \otimes gx^3. \end{aligned}$$

We have $\Delta(t)p_R = \Delta(x^3)\Delta(1-g)p_R$. Writing $\Phi = 1 - 2p_- \otimes p_- \otimes p_-$ under the form

$$\begin{aligned} \Phi &= \frac{3}{4}1 \otimes 1 \otimes 1 + \frac{1}{4}(1 \otimes 1 \otimes g + 1 \otimes g \otimes 1 + g \otimes 1 \otimes 1) \\ &\quad - \frac{1}{4}(1 \otimes g \otimes g + g \otimes 1 \otimes g + g \otimes g \otimes 1) + \frac{1}{4}g \otimes g \otimes g, \end{aligned}$$

one can easily see that $\Phi^{-1} = \Phi$ and that

$$p_R = x^1 \otimes x^2 \beta S(x^3) = x^1 \otimes x^2 x^3 = \frac{1}{2}(1 \otimes 1 + 1 \otimes g + g \otimes 1 - g \otimes g).$$

Now, using $\Delta(1-g)p_R = 1 \otimes 1 - g \otimes g$ we conclude that

$$\begin{aligned} \Delta(t)p_R &= (\bar{\omega} + \omega g)x^3 \otimes 1 + (\omega + \bar{\omega} g)x^3 \otimes g \pm ix^2 \otimes x + (\bar{\omega} g - \omega)x \otimes x^2 \\ &\quad + 1 \otimes x^3 \pm igx^2 \otimes gx - (\omega g - \bar{\omega})x \otimes gx^2 + g \otimes gx^3. \end{aligned}$$

Let now $\lambda = \sum_{i,j} c_{ij} P_{g^i x^j}$ be an element of H^* . It follows that λ satisfies (iii) in the statement of Theorem 3.7 if and only if $c_{01} = c_{11} = c_{13} = 0$, and the following relations hold

$$\bar{\omega}c_{12} - \omega c_{02} = 0, \quad \bar{\omega}c_{00} + \omega c_{10} = 0, \quad \bar{\omega}c_{02} - \omega c_{12} = 0 \quad \text{and} \quad \omega c_{00} + \bar{\omega}c_{10} = 0.$$

We find that $c_{00} = c_{02} = c_{10} = c_{12} = 0$, and so $\lambda = c_{03}P_{x^3}$. We thus have that $\mathcal{L} = kP_{x^3}$, as stated. \square

Applying Theorem 3.7 to H^{cop} , we find the following equivalent characterizations of right cointegrals.

Corollary 3.10. *Let H be a finite dimensional quasi-Hopf algebra and Λ a non-zero element of H^* . Then Λ is a right cointegral on H if and only if one of the equivalent relations below is satisfied:*

$$\Lambda(\tilde{q}^1 t_1 \tilde{p}^1) \tilde{q}^2 t_2 \tilde{p}^2 = \mu^{-1}(\alpha) \Lambda(t) 1, \quad \text{for all } t \in \int_l^H;$$

$$\Lambda(t_1 \tilde{p}^1) t_2 \tilde{p}^2 = \mu^{-1}(\alpha) \Lambda(t) S^{-1}(\alpha), \quad \text{for all } t \in \int_l^H;$$

$$\Lambda(ht_1 \tilde{p}^1) t_2 \tilde{p}^2 = \mu^{-1}(\alpha) \Lambda(t) S^{-1}(h\alpha), \quad \text{for all } t \in \int_l^H \text{ and } h \in H.$$

Examples of right cointegrals will be presented in the sequel, see Examples 4.7 and 4.8.

We end this section by recalling that in any finite dimensional quasi-Hopf algebra H we have

$$q^1 t_1 \otimes q^2 t_2 = \tilde{q}^1 t_1 \otimes \tilde{q}^2 t_2 \quad \text{and} \quad r_1 p^1 \otimes r_2 p^2 = r_1 \tilde{p}^1 \otimes r_2 \tilde{p}^2 \quad (3.21)$$

for all $t \in \int_l^H$ and $r \in \int_r^H$; the first formula appears in the proof of [4, Lemma 6.1] while the second one is the “op”-version of it. Thus in the equivalent conditions in Corollary 3.10, we can interchange q_R and q_L .

4. Integrals, cointegrals and the fourth power of the antipode

In this section we present several Frobenius systems for a finite dimensional quasi-Hopf algebra H in terms of integrals and cointegrals. Then we will see that the formula for the fourth power of the antipode S proved in [11,12] simplifies in some particular situations. First we recall some equivalent conditions for a finite dimensional k -algebra A to be Frobenius:

- A and A^* are isomorphic as right A -modules, where A^* is a right A -module via $\langle a^* \leftarrow a, b \rangle = a^*(ab)$, for all $a^* \in A^*$ and $a, b \in A$;
- A and A^* are isomorphic as left A -modules, where A^* is considered as a left A -module via the action $\langle a \rightarrow a^*, b \rangle = a^*(ba)$, for all $a^* \in A^*$ and $a, b \in A$;
- there exists a pair (ϕ, e) , called Frobenius pair or Frobenius system, with $\phi \in A^*$ and $e = e^1 \otimes e^2 \in A \otimes A$ (formal notation, summation implicitly understood), such that

$$ae^1 \otimes e^2 = e^1 \otimes e^2 a, \quad \forall a \in A, \quad \text{and} \quad \phi(e^1) e^2 = \phi(e^2) e^1 = 1.$$

The Frobenius system (ϕ, e) is unique in the following sense: any other Frobenius system for A is either of the form $(\phi \leftarrow d, e^1 \otimes d^{-1} e^2)$ or $(\mathbf{d} \rightarrow \phi, e^1 \mathbf{d}^{-1} \otimes e^2)$, for some suitable invertible elements d, \mathbf{d} of A . Moreover, if $(\psi, f = f^1 \otimes f^2)$ is another Frobenius system for A then

$$d = \psi(e^1) e^2, \quad d^{-1} = \phi(f^1) f^2, \quad \mathbf{d} = \chi^{-1}(d) \quad \text{and} \quad \mathbf{d}^{-1} = \chi^{-1}(d^{-1}), \quad (4.1)$$

where χ is the Nakayama automorphism associated to the Frobenius system (ϕ, e) . χ is the automorphism of A uniquely determined by the equality $a \rightarrow \phi = \phi \leftarrow \chi(a)$, satisfied for all $a \in A$. It is well known, see [12], that

$$\chi(a) = \phi(e^1 a) e^2 \quad \text{and} \quad \chi^{-1}(a) = \phi(a e^2) e^1, \quad (4.2)$$

for all $a \in A$. Finally, let $\{a_i\}_i$ be a basis of A with corresponding dual basis $\{a^i\}_i$ of A^* . If $f: A \rightarrow A^*$ is an isomorphism of right A -modules then $(f(1_A), \sum_i a_i \otimes f^{-1}(a^i))$ is a Frobenius system for A . Likewise, if $f: A \rightarrow A^*$ is a left A -linear isomorphism then $(f(1_A), \sum_i f^{-1}(a^i) \otimes a_i)$ is a Frobenius system for A .

We will now describe Frobenius systems for a finite dimensional quasi-Hopf algebra H . As we will see in the proof of Proposition 4.1, one of these systems already appeared in [12].

Proposition 4.1. Let H be a finite dimensional quasi-Hopf algebra and $(\lambda, t) \in \mathcal{L} \times \int_l^H$ satisfying $\lambda(S^{-1}(t)) = 1$. Then $(\lambda \circ S^{-1}, q^1 t_1 p^1 \otimes S(q^2 t_2 p^2))$ is a Frobenius system for H with Nakayama automorphism given by $\chi(h) = \mu(h_1) S^2(h_2)$, for all $h \in H$.

Proof. The properties of ξ defined in (3.11) show that H is a Frobenius algebra. On the other hand, as we mentioned in the proof of Theorem 3.7, the unique map $\lambda \in H^*$ satisfying $\lambda(q^2 t_2 p^2) q^1 t_1 p^1 = 1$ is a non-zero left cointegral on H . In terms of ξ this means that $\xi(\lambda \circ S^{-1}) = 1$, and so $\lambda \circ S^{-1}$ is a Frobenius morphism for H with Frobenius element

$$e = \sum_i \xi(e^i) \otimes e_i = q^1 t_1 p^1 \otimes S(q^2 t_2 p^2).$$

Thus $(\lambda \circ S^{-1}, q^1 t_1 p^1 \otimes S(q^2 t_2 p^2))$ is a Frobenius system for H , where $(\lambda, t) \in \mathcal{L} \times \int_l^H$ is such that $\lambda(q^2 t_2 p^2) q^1 t_1 p^1 = 1$ or, equivalently, $\lambda(S^{-1}(q^1 t_1 p^1)) q^2 t_2 p^2 = 1$.

Therefore, all we have to prove at this moment is that giving a pair $(\lambda, t) \in \mathcal{L} \times \int_l^H$ such that, for instance, $\lambda(S^{-1}(q^1 t_1 p^1)) q^2 t_2 p^2 = 1$ is equivalent to give a pair $(\lambda, t) \in \mathcal{L} \times \int_l^H$ such that $\lambda(S^{-1}(t)) = 1$. Indeed, if $\lambda(S^{-1}(q^1 t_1 p^1)) q^2 t_2 p^2 = 1$ then applying ε to the both sides we get $\lambda(S^{-1}(t)) = 1$.

Conversely, since S is an anti-algebra automorphism of H by [12, Lemma 3.1] we get that $(\lambda, q^2 t_2 p^2 \otimes S^{-1}(q^1 t_1 p^1))$ is another Frobenius system for H . Note that this Frobenius system already appeared in [12, Lemma 3.2]. From the uniqueness of the Frobenius system, it follows that there exists an invertible element \underline{g} in H such that

$$\lambda \circ S^{-1} = \lambda \leftarrow \underline{g} \quad \text{and} \quad q^1 t_1 p^1 \otimes S(q^2 t_2 p^2) = q^2 t_2 p^2 \otimes \underline{g}^{-1} S^{-1}(q^1 t_1 p^1). \quad (4.3)$$

Furthermore, by (4.1) we have

$$\underline{g} = \lambda(S^{-1}(q^2 t_2 p^2)) S^{-1}(q^1 t_1 p^1) \quad \text{and} \quad \underline{g}^{-1} = \lambda(q^1 t_1 p^1) S(q^2 t_2 p^2). \quad (4.4)$$

Thus, if $\lambda \in \mathcal{L}$ such that $\lambda(S^{-1}(t)) = 1$ then

$$1 = \lambda(S^{-1}(t)) = (\lambda \leftarrow \underline{g})(t) = \lambda(\underline{g}t) = \varepsilon(\underline{g})\lambda(t) = \mu(\beta)\lambda(S^{-1}(t))\lambda(t) = \mu(\beta)\lambda(t).$$

Then Theorem 3.7 says that $\lambda(q^2 t_2 p^2) q^1 t_1 p^1 = 1$ and as we already pointed out this is equivalent to $\lambda(S^{-1}(q^1 t_1 p^1)) q^2 t_2 p^2 = 1$.

To compute χ we use the formula in (4.2) to obtain that

$$\chi(h) = \phi(q^1 t_1 p^1 h) S(q^2 t_2 p^2) \stackrel{(3.19)}{=} \mu(h_1) \phi(q^1 t_1 p^1) S(q^2 t_2 p^2 S(h_2)) = \mu(h_1) S^2(h_2),$$

for all $h \in H$, or we can make use of (3.20). This finishes the proof. \square

The element \underline{g} defined in (4.4) is called the modular element of H . Together with μ , it plays an important role in the structure of the fourth power of the antipode. An equivalent version of this formula can be easily obtained as follows.

Remarks 4.2. (i) If $(\lambda, t) \in \mathcal{L} \times \int_l^H$ is such that $\lambda(S^{-1}(t)) = 1$, then the inverse of the map $\xi: H^* \rightarrow H$ considered in (3.11) is given by $\xi^{-1}(h) = h \rightharpoonup \lambda \circ S^{-1}$, for all $h \in H$. Indeed, ξ is left H -linear so $\xi(h \rightharpoonup \lambda \circ S^{-1}) = h\xi(\lambda \circ S^{-1}) = h\xi(\phi) = h$, for all $h \in H$. Thus the couple (λ, t) has also the property that $t \rightharpoonup \lambda \circ S^{-1} = \varepsilon$.

Note also that $\lambda \circ S^{-1} = \lambda \leftarrow \underline{g}$ implies that

$$\lambda \circ S = (\lambda \circ S^{-1} \leftarrow \underline{g}^{-1}) \circ S = S^{-1}(\underline{g}^{-1}) \rightharpoonup \lambda.$$

Furthermore, since $S(\underline{g}) = \lambda \circ S^{-2}(S(q^2 t_2 p^2)) q^1 t_1 p^1 = \xi(\lambda \circ S^{-2})$ we deduce that

$$\lambda \circ S^{-2} = \xi^{-1}(S(\underline{g})) = S(\underline{g}) \rightharpoonup \lambda \circ S^{-1} = S(\underline{g}) \rightharpoonup \lambda \leftarrow \underline{g}.$$

(ii) For all $h \in H$ we have that

$$\mu(f^1) S^{-2}(h) S^{-1}(\underline{g}^{-1}) S(f^2) = \mu(h_1 f^1) \mu^{-1}(h_{(2,2)}) S^{-1}(\underline{g}^{-1}) S(S(h_{(2,1)})) f^2, \quad (4.5)$$

where, as usual, f is the Drinfeld twist defined in (2.11). Observe that this equality can be viewed as an equivalent version of the formula for the fourth power of the antipode [11,12], namely

$$S^4(\mu^{-1} \rightharpoonup (h \leftarrow \mu)) = S^3(f_\mu^{-1}) S(\underline{g}) h S(\underline{g}^{-1}) S^3(f_\mu),$$

for all $h \in H$. Here $f_\mu := \mu(f^1) f^2$ and $h^* \rightharpoonup h := h^*(h_2) h_1$, for all $h^* \in H^*$ and $h \in H$.

To prove (4.5), we compute $\lambda(S^{-1}(h) q^1 t_1 p^1) q^2 t_2 p^2$ in two different ways. On one hand, by (2.21) we have

$$h q^1 t_1 \otimes q^2 t_2 = q^1 t_1 \otimes S^{-1}(h) q^2 t_2, \quad (4.6)$$

for all $h \in H$, and therefore $\lambda(S^{-1}(h) q^1 t_1 p^1) q^2 t_2 p^2 = \lambda(q^1 t_1 p^1) S^{-2}(h) q^2 t_2 p^2 = S^{-2}(h) S^{-1}(\underline{g})$, for all $h \in H$. On the other hand,

$$\begin{aligned} \lambda(S^{-1}(h) q^1 t_1 p^1) q^2 t_2 p^2 &\stackrel{(3,20)}{=} \mu(h_1) \lambda(q^1 t_1 p^1 S(h_2)) q^2 t_2 p^2 \\ &\stackrel{(3,19)}{=} \mu(h_1) \mu(S(h_2)_1) \lambda(q^1 t_1 p^1) q^2 t_2 p^2 S(S(h_2)_2) \\ &= \mu(h_1) \mu(S(h_2)_1) S^{-1}(\underline{g}^{-1}) S(S(h_2)_2). \end{aligned}$$

Comparing the two equalities above and using (2.8) we obtain (4.5).

Another Frobenius system for H can be obtained by working with H^{cop} instead of H . This will allow us to find a bijection between the spaces of left and right cointegrals on H .

Proposition 4.3. *Let H be a finite dimensional quasi-Hopf algebra, t a non-zero left integral in H and let $\lambda \in \mathcal{L}$ and $\Lambda \in \mathcal{R}$ be such that $\lambda(S^{-1}(t)) = 1$ and $\Lambda(S(t)) = 1$, respectively. Then $u := \mu(V^1) S^2(V^2)$ is invertible in H and $\lambda \circ S^{-1} = \Lambda \leftarrow u$. Consequently the map $\mathcal{L} \rightarrow \mathcal{R}$ sending λ to $\lambda \circ S^{-1} \leftarrow u^{-1}$ is a well-defined bijection.*

Proof. Applying Proposition 4.1 with H replaced by H^{cop} , we find that $(\Lambda \circ S, \tilde{q}^2 t_2 \tilde{p}^2 \otimes S^{-1}(\tilde{q}^1 t_1 \tilde{p}^1))$, with Λ the unique right cointegral on H such that $\Lambda(S(t)) = 1$, is a Frobenius system for H . By [12, Lemma 3.1] we have that $(\Lambda, \tilde{q}^1 t_1 \tilde{p}^1 \otimes S(\tilde{q}^1 t_2 \tilde{p}^1))$ is also a Frobenius system for H^{cop} , and therefore for H as well. Now, we know that the Frobenius systems $(\lambda \circ S^{-1}, q^1 t_1 p^1 \otimes S(q^2 t_2 p^2))$ and $(\Lambda, \tilde{q}^1 t_1 \tilde{p}^1 \otimes S(\tilde{q}^2 t_2 \tilde{p}^2))$ are related through an invertible element $u \in H$ satisfying

$$\lambda \circ S^{-1} = \Lambda \leftarrow u, \quad q^1 t_1 p^1 \otimes S(q^2 t_2 p^2) = \tilde{q}^1 t_1 \tilde{p}^1 \otimes u^{-1} S(\tilde{q}^2 t_2 \tilde{p}^2). \quad (4.7)$$

In order to prove the first assertion, it therefore suffices to show that $u = \mu(V^1) S^2(V^2)$. To this end, we first apply (4.1) and obtain that $u = \lambda(S^{-1}(\tilde{q}^1 t_1 \tilde{p}^1)) S(\tilde{q}^2 t_2 \tilde{p}^2)$. Using (2.15) we compute

$$\begin{aligned}
f^1 \tilde{p}^1 \otimes f^2 \tilde{p}^2 &\stackrel{(2.18)}{=} f^1 X^2 S^{-1}(X^1 \beta) \otimes f^2 X^3 \\
&\stackrel{(2.15), (2.5)}{=} f^1 X^2 g_2^1 G^2 \alpha S^{-1}(X^1 g_1^1 G^1) \otimes f^2 X^3 g^2 \\
&\stackrel{(2.14)}{=} f^1 g_1^2 G^1 S(X^2) \alpha X^3 S^{-1}(g^1) \otimes f^2 g_2^2 G^2 S(X^1) \\
&\stackrel{(2.8), (2.17)}{=} S(q^2 S^{-1}(g^2)_2) S^{-1}(g^1) \otimes S(q^1 S^{-1}(g^2)_1),
\end{aligned}$$

where $f = f^1 \otimes f^2 = F^1 \otimes F^2$ and $f^{-1} = g^1 \otimes g^2 = G^1 \otimes G^2$. From (2.20) it follows that any right integral r in H satisfies

$$r_1 p^1 h \otimes r_2 p^2 = r_1 p^1 \otimes r_2 p^2 S(h), \quad (4.8)$$

for all $h \in H$. In particular, if we set $r := S^{-1}(t) \in \int_r^H$ then

$$\begin{aligned}
t_1 \tilde{p}^1 \otimes t_2 \tilde{p}^2 &\stackrel{(2.8)}{=} g^1 S(r_2) f^1 \tilde{p}^1 \otimes g^2 S(r_1) f^2 \tilde{p}^2 \\
&= g^1 S(q^2 S^{-1}(G^2)_2 r_2) S^{-1}(G^1) \otimes g^2 S(q^1 S^{-1}(G^2)_1 r_1) \\
&\stackrel{(3.9), (2.12)}{=} \mu(G^2) \alpha_1 \delta^1 S(q^2 r_2) S^{-1}(G^1) \otimes \alpha_2 \delta^2 S(q^1 r_1) \\
&\stackrel{(2.10)}{=} \mu(G^2) \alpha_1 \beta S(q^2 r_2 X^3) S^{-1}(G^1) \otimes \alpha_2 X^1 \beta S(q^1 r_1 X^2) \\
&\stackrel{(2.18), (3.21)}{=} \mu(G^2) \alpha_1 \beta S(q^2 r_2 p^2) S^{-1}(G^1) \otimes \alpha_2 S(q^1 r_1 p^1) \\
&\stackrel{(4.8)}{=} \mu(G^2) \alpha_1 \beta S(q^2 r_2 p^2 \alpha_2) S^{-1}(G^1) \otimes S(q^1 r_1 p^1) \\
&\stackrel{(2.5)}{=} \mu(G^2) \beta S(q^2 r_2 p^2) S^{-1}(G^1) \otimes S(q^1 r_1 p^1),
\end{aligned}$$

and therefore

$$\begin{aligned}
\tilde{q}^1 t_1 \tilde{p}^1 \otimes \tilde{q}^2 t_2 \tilde{p}^2 &= \mu(G^2) \tilde{q}^1 \beta S(q^2 r_2 p^2) S^{-1}(G^1) \otimes \tilde{q}^2 S(q^1 r_1 p^1) \\
&\stackrel{(4.8)}{=} \mu(G^2) \tilde{q}^1 \beta S(q^2 r_2 p^2 \tilde{q}^2) S^{-1}(G^1) \otimes S(q^1 r_1 p^1) \\
&\stackrel{(2.19), (2.6)}{=} \mu(G^2) S(q^2 r_2 p^2) S^{-1}(G^1) \otimes S(q^1 r_1 p^1).
\end{aligned}$$

Then we compute that

$$\begin{aligned}
u &= \mu(G^2) \lambda(S^{-2}(G^1) q^2 r_2 p^2) S^2(q^1 r_1 p^1) \\
&\stackrel{(3.20)}{=} \mu(g^2) \mu(S^{-1}(g^1)_1) \lambda(q^2 r_2 p^2 S(S^{-1}(g^1)_2)) S^2(q^1 r_1 p^1) \\
&\stackrel{(4.8)}{=} \mu(g^2) \mu(S^{-1}(g^1)_1) \lambda(q^2 r_2 p^2) S^2(q^1 r_1 p^1 S^{-1}(g^1)_2).
\end{aligned}$$

Using the fact that r is right integral in H , we find that

$$\lambda(q^2 r_2 p^2) q^1 r_1 p^1 \stackrel{(3.17)}{=} \mu(x^1) \lambda(S^{-1}(\tilde{q}^1) r S(x^2 \tilde{p}^1)) \tilde{q}^2 x^3 \tilde{p}^2 \stackrel{(3.9)}{=} \lambda(S^{-1}(t)) \tilde{q}^2 = \mu(\tilde{q}^1) \tilde{q}^2,$$

and then we can finally compute that

$$\begin{aligned}
u &= \mu(g^2 \tilde{q}^1) \mu(S^{-1}(g^1)_1) S^2(\tilde{q}^2 S^{-1}(g^1)_2) \\
&\stackrel{(2.19), (2.8)}{=} \mu(g^2 S(x^1) \alpha S^{-1}(f^2 g_2^1 G^2 S(x^2))) S(f^1 g_1^1 G^1 S(x^3)) \\
&\stackrel{(2.14), (2.15)}{=} \mu(x^3 S^{-1}(f^2 x^2 \beta)) S(f^1 x^1) \stackrel{(2.17)}{=} \mu^{-1}(f^2 p^2) S(f^1 p^1) \stackrel{(3.5)}{=} \mu(V^1) S^2(V^2),
\end{aligned}$$

as needed. Using (4.1) or a simple observation, we see that

$$u^{-1} = \mu^{-1}(q_2^1 g^2 S(q^2)) S(q_1^1 g^1). \quad (4.9)$$

From the uniqueness of left and right cointegrals on H , it follows that the map $\mathcal{L} \rightarrow \mathcal{R}$ in the statement is well defined. Its inverse sends $\Lambda \in \mathcal{R}$ to $(\Lambda \leftarrow u) \circ S \in \mathcal{L}$. \square

Corollary 4.4. *If H is a finite dimensional unimodular quasi-Hopf algebra and \bar{S} is the antipode of H^* then $\bar{S}^{-1}(\mathcal{L}) = \mathcal{R}$.*

Proof. $\mu = \varepsilon$ implies $u = 1$, and so $\lambda \circ S^{-1} = \Lambda$. Everything then follows from the uniqueness of left and right cointegrals on H . \square

Corollary 4.5. *Consider $(\Lambda, t) \in \mathcal{R} \times \int_1^H$ such that $\Lambda(S(t)) = 1$. Then*

$$\Lambda \circ S = \Lambda \leftarrow u S^2(S^{-1}(u^{-1}) \leftarrow \mu) \underline{g}^{-1},$$

where \underline{g} is the modular element of H , and where we define $h \leftarrow h^* := h^*(h_1)h_2$, for all $h^* \in H^*$ and $h \in H$.

Proof. If $(\lambda, t) \in \mathcal{L} \times \int_1^H$ satisfies $\lambda(S^{-1}(t)) = 1$ then we know that $\lambda \circ S^{-1} = \lambda \leftarrow \underline{g}$. Now we write this property in H^{cop} : for any couple $(\Lambda, t) \in \mathcal{R} \times \int_1^H$ such that $\Lambda(S(t)) = 1$, we have that $\Lambda \circ S = \Lambda \leftarrow \underline{g}_{\text{cop}}$ with $\underline{g}_{\text{cop}} = \Lambda(S(\tilde{q}^1 t_1 \tilde{p}^1)) S(\tilde{q}^2 t_2 \tilde{p}^2)$. The proof is finished after we show that $\underline{g}_{\text{cop}} = u S^2(S^{-1}(u^{-1}) \leftarrow \mu) \underline{g}^{-1}$.

By $\lambda \circ S^{-1} = \Lambda \leftarrow u$ we get $\Lambda \circ S = (\lambda \circ S^{-1} \leftarrow u^{-1}) \circ S = S^{-1}(u^{-1}) \rightarrow \lambda$, so

$$\begin{aligned}
\underline{g}_{\text{cop}} &= \lambda(\tilde{q}^1 t_1 \tilde{p}^1 S^{-1}(u^{-1})) S(\tilde{q}^2 t_2 \tilde{p}^2) \\
&\stackrel{(4.7)}{=} \lambda(q^1 t_1 p^1 S^{-1}(u^{-1})) u S(q^2 t_2 p^2) \\
&\stackrel{(3.19)}{=} \mu(S^{-1}(u^{-1})_1) \lambda(q^1 t_1 p^1) u S^2(S^{-1}(u^{-1})_2) S(q^2 t_2 p^2) \\
&\stackrel{(4.4)}{=} u S^2(S^{-1}(u^{-1}) \leftarrow \mu) \underline{g}^{-1},
\end{aligned}$$

as desired. Notice that $\underline{g}_{\text{cop}} = \underline{g}^{-1}$ in the case when $u = 1$, and this happens for instance when H is unimodular. \square

Corollary 4.6. *Let H be a finite dimensional quasi-Hopf algebra and assume that $\mathcal{L} = \mathcal{R}$. Then $\underline{g} = \mu(\beta) \mu^{-1}(\beta)^{-1} u$. Consequently, if H is unimodular and admits a non-zero left cointegral that is at the same time right cointegral then $\underline{g} = 1$.*

Proof. Since $\dim_k \mathcal{L} = \dim_k \mathcal{R} = 1$ it follows that $\mathcal{L} = \mathcal{R}$ if and only if $\mathcal{L} \cap \mathcal{R} \neq 0$.

Let $0 \neq \lambda \in \mathcal{L} \cap \mathcal{R}$ and $t \in \int_1^H$ such that $\lambda(S^{-1}(t)) = 1$. Then $\lambda \circ S^{-1} = \Lambda \leftarrow u$, for some non-zero $\Lambda \in \mathcal{R}$. But $\Lambda = c\lambda$ for a certain $c \in k$, so $\lambda \circ S^{-1} = c\lambda \leftarrow u$. We have $\mu(\beta)\lambda(t) = 1$, hence

$$1 = \lambda(S^{-1}(t)) = c\lambda(ut) = c\varepsilon(u)\lambda(t) = c\mu^{-1}(\beta)\mu(\beta)^{-1}.$$

We get $c = \mu(\beta)\mu^{-1}(\beta)^{-1}$ and therefore, using (4.6), we conclude that

$$\begin{aligned}\underline{g} &= \lambda(S^{-1}(q^2 t_2 p^2))S^{-1}(q^1 t_1 p^1) = c\lambda(uq^2 t_2 p^2)S^{-1}(q^1 t_1 p^1) \\ &\stackrel{(4.6)}{=} c\lambda(q^2 t_2 p^2)S^{-1}(q^1 t_1 p^1)u = \mu(\beta)\mu^{-1}(\beta)^{-1}u,\end{aligned}$$

as stated. \square

The above formulas tell us how to find right cointegrals from left cointegrals, and vice-versa.

Example 4.7. For $H(2)$ we have that P_g is at the same time left and right cointegral and $\underline{g} = 1$.

Proof. $H(2)$ is unimodular and has the antipode defined by the identity map. Thus in this particular case the formula $\lambda \circ S^{-1} = \Lambda \leftarrow u$ reduces to $\lambda = \Lambda$, and so $\mathcal{L} = \mathcal{R}$. From Example 3.8 we deduce that P_g is a left and right non-zero cointegral on $H(2)$, and from Corollary 4.6 we get $\underline{g} = 1$. \square

Example 4.8. For $H_{\pm}(8)$ we have $\mathcal{R} = k(\omega P_{x^3} + \bar{\omega} P_{gx^3})$, $\underline{g} = \omega 1 + \bar{\omega} g$ and $\underline{g}^{-1} = \bar{\omega} 1 + \omega g$, respectively.

Proof. To find a right cointegral on $H_{\pm}(8)$ we compute $\lambda \circ S^{-1}$ and the element u . Then $\lambda \circ S^{-1} \leftarrow u^{-1}$ will be a non-zero right cointegral on $H_{\pm}(8)$.

Consider the left integral $t = (1 + g)x^3$ and take $\lambda = cP_{x^3}$ with $c \in k$ that has to be determined such that $\lambda(S^{-1}(t)) = 1$. Actually, since $\beta = 1$ we need to find that unique $c \in k$ such that $\lambda(t) = 1$ and it then follows that we should have $c = 1$, and thus $\lambda = P_{x^3}$.

We use now $(p_+ \pm ip_-)(p_+ \mp ip_-) = 1$ to see that $S^{-1}(x) = -(p_+ \mp ip_-)x$, and

$$\begin{aligned}S^{-1}(x^2) &= \mp ix^2, & S^{-1}(x^3) &= \pm i(p_+ \mp ip_-)x^3, & S^{-1}(gx) &= (p_+ \pm ip_-)x, \\ S^{-1}(gx^2) &= \mp igx^2, & S^{-1}(gx^3) &= \mp i(p_+ \pm ip_-)x^3.\end{aligned}$$

In particular, we get $\lambda(S^{-1}(g^i x^j)) = 0$ unless in the following two cases when we have

$$\begin{aligned}\lambda(S^{-1}(x^3)) &= \pm i\lambda(p_+ x^3) + \lambda(p_- x^3) = \frac{1}{2}(1 \pm i) = \omega, \quad \text{and respectively} \\ \lambda(S^{-1}(gx^3)) &= \mp i\lambda(p_+ x^3) + \lambda(p_- x^3) = \frac{1}{2}(\mp i + 1) = \bar{\omega}.\end{aligned}$$

In other words, we have $\lambda \circ S^{-1} = \omega P_{x^3} + \bar{\omega} P_{gx^3}$. It can be easily checked that $f = f^{-1} = p_R$ in the case when $H = H_{\pm}(8)$. We conclude that $u = 1$, even if H is not unimodular. Thus $\omega P_{x^3} + \bar{\omega} P_{gx^3}$ is a right non-zero cointegral on $H_{\pm}(8)$.

Let us finally compute \underline{g} . Since $\beta = 1$ formula (3.18) implies $q^2 t_2 p^2 \otimes q^1 t_1 p^1 = t_1 p^1 \otimes t_2 p^2$, hence $\underline{g} = \lambda(S^{-1}(t_2 p^2))S^{-1}(t_1 p^1)$. By the expression of $\Delta(t)p_R$ found in Example 3.9 we then obtain

$$\underline{g} = \lambda(S^{-1}(x^3))1 + \lambda(S^{-1}(gx^3))g = \omega 1 + \bar{\omega} g,$$

as desired. A simple inspection shows that $\underline{g}^{-1} = \bar{\omega} 1 + \omega g$, and this completes the proof. \square

We now investigate the relation between $\lambda \circ S$ and Λ .

Proposition 4.9. Let t be a left integral in H and let $\lambda \in \mathcal{L}$ and $\Lambda \in \mathcal{R}$ be such that $\lambda(S^{-1}(t)) = 1$ and $\Lambda(S(t)) = 1$. Then $v = (\mu^{-1}(g)\mu(\beta))^{-1}\mu(S(p^2)f^1)S(p^1)f^2$ is invertible in H and $\lambda \circ S = \Lambda \leftarrow v$. Consequently, we have a well-defined bijection between \mathcal{L} and \mathcal{R} mapping λ to $\lambda \circ S \leftarrow v^{-1}$.

Proof. Applying Proposition 4.1 to H^{op} , we find that $(\phi_{\text{op}}, q^1 r_1 p^1 \otimes S^{-1}(q^2 r_2 p^2))$ is a Frobenius system for H ; r is a non-zero right integral in H , and ϕ_{op} is the unique element of H^* satisfying $\phi_{\text{op}}(S^{-1}(q^2 r_2 p^2)) q^1 r_1 p^1 = 1$ or, equivalently, $\phi_{\text{op}}(q^1 r_1 p^1) q^2 r_2 p^2 = 1$. Now set $t = S(r) \in \int_l^H$ and take $\lambda \in \mathcal{L}$ and $\Lambda \in \mathcal{R}$ such that $\lambda(S^{-1}(t)) = 1$ and $\Lambda(S(t)) = 1$. We will prove that $\phi_{\text{op}} = \mu^{-1}(\tilde{p}^1) S^{-2}(\tilde{p}^2) \rightharpoonup \lambda \circ S$. To this end recall from the proof of [4, Lemma 6.1] that

$$V^1 r_1 U^1 \otimes V^2 r_2 U^2 = S^{-1}(q^2 t_2 p^2) \otimes S^{-1}(q^1 t_1 p^1), \quad (4.10)$$

$$U = \tilde{q}_1^1 p^1 \otimes \tilde{q}_2^2 p^2 S(\tilde{q}^2) \quad \text{and} \quad V = q^1 \tilde{p}_1^1 \otimes S^{-1}(\tilde{p}^2) q^2 \tilde{p}_2^1. \quad (4.11)$$

From (4.11) it follows that

$$\begin{aligned} V^1 r_1 U^1 \otimes V^2 r_2 U^2 &= q^1 \tilde{p}_1^1 r_1 \tilde{q}_1^1 p^1 \otimes S^{-1}(\tilde{p}^2) q^2 \tilde{p}_2^1 r_2 \tilde{q}_2^2 p^2 S(\tilde{q}^2) \\ &= \mu^{-1}(\tilde{p}^1) q^1 r_1 p^1 \otimes S^{-1}(\tilde{p}^2) q^2 r_2 p^2, \end{aligned}$$

and then (4.10) can be restated as

$$\mu^{-1}(\tilde{p}^1) q^1 r_1 p^1 \otimes S^{-1}(\tilde{p}^2) q^2 r_2 p^2 = S^{-1}(q^2 t_2 p^2) \otimes S^{-1}(q^1 t_1 p^1).$$

Observe now that, for all $h \in H$,

$$h q^1 r_1 \otimes q^2 r_2 \stackrel{(2.21)}{=} q^1 h_{(1,1)} r_1 \otimes S^{-1}(h_2) q^2 h_{(1,2)} r_2 \stackrel{(3.9)}{=} \mu^{-1}(h_1) q^1 r_1 \otimes S^{-1}(h_2) q^2 r_2.$$

Since

$$\mu(\tilde{q}^1) \mu^{-1}(\tilde{p}^1) \tilde{q}^2 q^1 r_1 p^1 \otimes S^{-1}(\tilde{p}^2) q^2 r_2 p^2 = \mu(\tilde{q}^1) S^{-1}(q^2 t_2 p^2 S(\tilde{q}^2)) \otimes S^{-1}(q^1 t_1 p^1),$$

this implies that

$$q^1 r_1 p^1 \otimes q^2 r_2 p^2 = \mu(\tilde{q}^1) \tilde{q}^2 S^{-1}(q^2 t_2 p^2) \otimes S^{-1}(q^1 t_1 p^1), \quad (4.12)$$

where we made use of (2.27).

In the proof of Proposition 4.3 we have shown that $\lambda(q^2 r_2 p^2) q^1 r_1 p^1 = \mu(\tilde{q}^1) \tilde{q}^2$, hence

$$\begin{aligned} 1 &\stackrel{(2.26)}{=} \mu(S(\tilde{p}^1) \tilde{q}^1 \tilde{p}_1^2) \tilde{q}^2 \tilde{p}_2^2 = \mu^{-1}(\tilde{p}^1) \mu(\tilde{p}_1^2) \lambda(q^2 r_2 p^2) q^1 r_1 p^1 \tilde{p}_2^2 \\ &\stackrel{(4.8)}{=} \mu^{-1}(\tilde{p}^1) \mu(\tilde{p}_1^2) \lambda(q^2 r_2 p^2 S(\tilde{p}_2^2)) q^1 r_1 p^1 \\ &\stackrel{(3.20)}{=} \mu^{-1}(\tilde{p}^1) \lambda(S^{-1}(\tilde{p}^2) q^2 r_2 p^2) q^1 r_1 p^1 \\ &= \langle \lambda \leftarrow \mu^{-1}(\tilde{p}^1) S^{-1}(\tilde{p}^2), q^2 r_2 p^2 \rangle q^1 r_1 p^1. \end{aligned}$$

From the uniqueness of the map ϕ_{op} it follows that

$$\phi_{\text{op}} = (\lambda \leftarrow \mu^{-1}(\tilde{p}^1) S^{-1}(\tilde{p}^2)) \circ S = \mu^{-1}(\tilde{p}^1) S^{-2}(\tilde{p}^2) \rightharpoonup \lambda \circ S,$$

as we claimed. By the definition of p_L it is immediate that $d := \mu^{-1}(\tilde{p}^1) S^{-2}(\tilde{p}^2)$ is invertible in H , and therefore $(\lambda \circ S, q^1 r_1 p^1 d \otimes S^{-1}(q^2 r_2 p^2))$ is a Frobenius system for H , whenever $(\lambda, r) \in \mathcal{L} \times \int_r^H$ is such that $\lambda(r) = 1$. Comparing it with $(\Lambda, \tilde{q}^1 t_1 \tilde{p}^1 \otimes S(\tilde{q}^2 t_2 \tilde{p}^2))$ we conclude that there is an invertible

element $v \in H$ such that $\lambda \circ S = \Lambda \leftarrow v$. In order to compute v , observe first that (2.19), (2.14) and the formula $S^{-1}(f^2)\beta f^1 = S^{-1}(\alpha)$ imply

$$S(\tilde{p}^2)f^1 \otimes S(\tilde{p}^1)f^2 = q^1 g_1^1 \otimes S^{-1}(g^2)q^2 g_2^1, \quad (4.13)$$

where, as usual, we denote $f^{-1} = g^1 \otimes g^2$.

According to (4.1) we have that

$$\begin{aligned} v &= \lambda(S(\tilde{q}^1 t_1 \tilde{p}^1))S(\tilde{q}^2 t_2 \tilde{p}^2) \\ &\stackrel{(2.8), (3.21)}{=} \langle \lambda, S(\tilde{p}^1)f^2 S(t)_2 g^2 S(q^1) \rangle S(\tilde{p}^2)f^1 S(t)_1 g^1 S(q^2) \\ &\stackrel{(4.13), (3.5)}{=} \langle \lambda, S^{-1}(g^2)q^2 (g^1 S(t))_2 U^2 \rangle q^1 (g^1 S(t))_1 U^1 \\ &\stackrel{(3.20), (3.9)}{=} \mu(g_1^2)\mu^{-1}(g^1) \langle \lambda, q^2 S(t)_2 U^2 S(g_2^2) \rangle q^1 S(t)_1 U^1 \\ &\stackrel{(3.12), (*)}{=} \mu(g_1^2)\mu^{-1}(g^1) \langle \lambda, q^2 S(t)_2 U^2 \rangle q^1 S(t)_1 U^1 g_2^2 \\ &\stackrel{(3.14), (3.9)}{=} \mu^{-1}(g^1)\mu(\tilde{q}^1 g_1^1) \langle \lambda, V^2 S(t)_2 U^2 \rangle \tilde{q}^2 V^1 S(t)_1 U^1 g_2^2 \\ &\stackrel{(3.6), (*)}{=} \mu(S(g^1)\tilde{q}^1 g_1^1) \lambda(S(t)) \tilde{q}^2 g_2^2. \end{aligned}$$

At (*) we used that $S(t) \in J_r^H$. Now we claim that

$$S(g^1)\tilde{q}^1 g_1^2 \otimes \tilde{q}^2 g_2^2 = S(p^2)f^1 \otimes S(p^1)f^2. \quad (4.14)$$

Indeed, using (2.19) and the op-version of (2.14), we compute

$$\begin{aligned} S(g^1)\tilde{q}^1 g_1^2 \otimes \tilde{q}^2 g_2^2 &= S(g_1^1 G^1 S(x^3))\alpha g_2^1 G^2 S(x^2)f^1 \otimes g^2 S(x^1)f^2 \\ &\stackrel{(2.5), (2.15)}{=} S(x^2 \beta S(x^3))f^1 \otimes S(x^1)f^2 = S(p^2)f^1 \otimes S(p^1)f^2. \end{aligned}$$

We also have that $\lambda(S(t)) = (S^{-1}(\underline{g}^{-1}) \rightharpoonup \lambda)(t) = \mu^{-1}(\underline{g}^{-1})\lambda(t) = (\mu^{-1}(\underline{g})\mu(\beta))^{-1}$, hence

$$v = (\mu^{-1}(\underline{g})\mu(\beta))^{-1} \mu(S(p^2)f^1)S(p^1)f^2.$$

It is easy to see that the inverse of v is

$$v^{-1} = \mu^{-1}(\underline{g})\mu(\beta q^2 g^1 S(q_2^1))g^2 S(q_1^1),$$

and this finishes the proof. \square

Corollary 4.10. Let \bar{S} be the antipode of the dual of a finite dimensional unimodular quasi-Hopf algebra H . Then $\bar{S}(\mathcal{L}) = \mathcal{R}$.

Proof. In this case we have $\mu = \varepsilon$ and therefore $v = 1$. \square

For $a \in H$ invertible, let Inn_a be the inner automorphism of H defined by a , this means that $\text{Inn}_a(h) = aha^{-1}$, for all $h \in H$.

Corollary 4.11. *If H is a finite dimensional unimodular quasi-Hopf algebra then $S^4 = \text{Inn}_{S(\underline{g})}$, where \underline{g} is the modular element of H . Furthermore, if $\mathcal{L} = \mathcal{R}$ then S^4 is the identity morphism of H .*

Proof. If H is unimodular, then $\mu = \varepsilon$ and $f_\mu = 1$ and if, moreover, $\mathcal{L} = \mathcal{R}$ then $\underline{g} = 1$, cf. Corollary 4.6. \square

We have seen how the antipode of H^* , or its inverse, acts on the space of left or right cointegrals. Let us now investigate how the antipode of H acts on the space of left or right integrals in H .

Proposition 4.12. *If t , respectively r , are non-zero left, respectively right, integrals in H then*

$$\begin{aligned} S(t) &= \mu(\beta)^{-1} \mu(q^2 t_2 p^2) q^1 t_1 p^1; \\ S^{-1}(t) &= \mu^{-1}(\underline{g}) \mu(q^2 t_2 p^2) q^1 t_1 p^1; \\ S(r) &= (\mu^{-1}(\underline{g}) \mu(\alpha \beta))^{-1} \mu^{-1}(q^2 r_2 p^2) q^1 r_1 p^1; \\ S^{-1}(r) &= \mu(\alpha)^{-1} \mu^{-1}(q^2 r_2 p^2) q^1 r_1 p^1. \end{aligned}$$

Proof. Consider $\lambda \in \mathcal{L}$ such that $\lambda(S^{-1}(t)) = 1$, so that $\mu(\beta)\lambda(t) = 1$. If we define $r' = \mu(t_2 p^2) t_1 p^1$ then

$$\begin{aligned} r'h &= \mu(t_2 p^2) t_1 p^1 h \stackrel{(2.20)}{=} \mu(t_2 h_{(1,2)} p^2 S(h_2)) t_1 h_{(1,1)} p^1 \\ &\stackrel{(3.9)}{=} \mu(h_1) \mu(t_2 p^2) \mu^{-1}(h_2) t_1 p^1 = \varepsilon(h) r', \end{aligned}$$

for all $h \in H$. Thus r' is a right integral in H . As $\dim_k \int_r^H = 1$ there exist $c, c' \in k$ such that $S(t) = cr'$ and $S^{-1}(t) = c'r'$. We have that $\lambda(S(t)) = (S^{-1}(\underline{g}^{-1}) \rightharpoonup \lambda)(t) = \mu^{-1}(\underline{g}^{-1})\lambda(t) = (\mu^{-1}(\underline{g})\mu(\beta))^{-1}$ and

$$\lambda(r') = \mu(t_2 p^2) \lambda(t_1 p^1) \stackrel{(3.18)}{=} \mu(S^{-1}(\beta) q^2 t_2 p^2) \lambda(q^1 t_1 p^1) \stackrel{(4.4)}{=} \mu^{-1}(\beta) \mu^{-1}(\underline{g}^{-1}).$$

It follows that $c = (\mu(\beta)\mu^{-1}(\beta))^{-1}$ and $c' = \mu^{-1}(\beta \underline{g}^{-1})^{-1}$. The formulas for $S(t)$ and $S^{-1}(t)$ now follow from these relations and (3.18).

Let r be a right integral, and let $t = S(r)$. Take $\lambda \in \mathcal{L}$ such that $\lambda(r) = \lambda(S^{-1}(t)) = 1$. Proceeding as in the first part of the proof, we can show that $t' := \mu^{-1}(q^2 r_2) q^1 r_1$ is a left integral, hence there exist $b, b' \in k$ such that $S(r) = bt'$ and $S^{-1}(r) = b't'$. The formula for $S^{-1}(r)$ can be obtained by applying the formula for $S(t)$ to H^{op} .

As the formula for $S^{-1}(t)$ contains \underline{g} and we do not have an analogue of \underline{g} in H^{op} we cannot derive the formula for $S(r)$ from the one of $S^{-1}(t)$. Nevertheless, we can obtain it by computing b as follows. We have

$$\begin{aligned} \lambda(S(r)) &= (S^{-1}(\underline{g}^{-1}) \rightharpoonup \lambda)(r) = \varepsilon(\underline{g}^{-1})\lambda(r) = \lambda(t) = \mu(\beta)^{-1}; \\ \lambda(t') &= \lambda(q^1 r_1) \mu^{-1}(q^2 r_2) \stackrel{(*)}{=} \lambda(q^1 r_1 p^1) \mu^{-1}(q^2 r_2 p^2 \alpha) \\ &\stackrel{(4.12)}{=} \mu(\tilde{q}^1) \lambda(\tilde{q}^2 S^{-1}(q^2 t_2 p^2)) \mu^{-1}(S^{-1}(q^1 t_1 p^1) \alpha) \\ &= \mu(\tilde{q}^1) \phi(q^2 t_2 p^2 S(\tilde{q}^2)) \mu(q^1 t_1 p^1) \mu^{-1}(\alpha). \end{aligned}$$

At $(*)$, we used (3.18), applied to H^{op} , and $\phi = \lambda \circ S^{-1}$ is the Frobenius morphism of H . We now use the Nakayama automorphism χ of H associated to ϕ to compute

$$\begin{aligned}
\lambda(t') &= \mu(\tilde{q}^1)\phi(\chi(S(\tilde{q}^2))q^2t_2p^2)\mu(q^1t_1p^1)\mu^{-1}(\alpha) \\
&\stackrel{(4.6)}{=} \mu(\tilde{q}^1)\phi(q^2t_2p^2)\mu(S(\chi(S(\tilde{q}^2)))q^1t_1p^1)\mu^{-1}(\alpha) \\
&= \mu(\tilde{q}^1)\lambda(S^{-1}(q^2t_2p^2))\mu^{-1}(S^{-1}(q^1t_1p^1))\mu^{-1}(\chi(S(\tilde{q}^2)))\mu^{-1}(\alpha) \\
&\stackrel{(4.4)}{=} \mu(\tilde{q}^1)\mu^{-1}(\underline{g})\mu^{-1}(S_\mu(\tilde{q}^2))\mu^{-1}(\alpha).
\end{aligned}$$

In the last equality, we used the (obvious) identity $\chi \circ S = S^2 \circ S_\mu$, and we used the notation $S_\mu(h) := S(h) \leftarrow \mu$. Now we have for all $h \in H$ that $\mu^{-1}(S_\mu(h)) = \mu(S(h)_1)\mu^{-1}(S(h)_2) = \varepsilon(S(h)) = \varepsilon(h)$, so $\lambda(t') = \mu(\alpha)\mu^{-1}(\underline{g}\alpha)$, and therefore $\mu(\beta)^{-1} = b\mu(\alpha)\mu^{-1}(\underline{g}\alpha)$. Hence

$$S(r) = \mu(\alpha\beta)^{-1}\mu^{-1}(\underline{g}\alpha)^{-1}\mu^{-1}(q^2r_2)q^1r_1 = (\mu^{-1}(\underline{g})\mu(\alpha\beta))^{-1}\mu^{-1}(q^2r_2p^2)q^1r_1p^1,$$

and this completes the proof. \square

Applying Proposition 4.12 and (3.10), we have the following result.

Corollary 4.13. *Let H be a finite dimensional quasi-Hopf algebra. For all $t \in \int_1^H$ and $r \in \int_r^H$, we have that*

$$S^2(t) = (\mu^{-1}(\underline{g})\mu(\beta))^{-1}t \quad \text{and} \quad S^2(r) = (\mu^{-1}(\underline{g})\mu(\beta))^{-1}r.$$

5. The (co)integrals of a quantum double

We are now able to provide explicit formulas for the integral in and the cointegral on the quantum double $D(H)$ of a finite dimensional quasi-Hopf algebra H . In particular, we will see that the conjecture of Hausser and Nill holds if H is unimodular, but not in general. More precisely, if H is finite dimensional, λ is a non-zero left cointegral, and r is a non-zero right integral, then $\mu^{-1}(\delta^2)\delta^1 \rightarrow \lambda \bowtie r$ is a non-zero left integral (Proposition 5.1) and a non-zero right integral (Proposition A.3) in $D(H)$. Recall that μ is the modular element of H^* , see (3.8), and δ is Drinfeld's element, see (2.10). This implies that the Drinfeld double of a finite dimensional quasi-Hopf algebra H is unimodular. Moreover, if H itself is unimodular, then $\mu^{-1}(\delta^2)\delta^1 = \beta$, and then $\beta \rightarrow \lambda \bowtie r$ is a non-zero left integral, proving in this case the Hausser–Nill conjecture announced at the end of [11].

We first recall the definition and properties of $D(H)$. In the sequel, $\{e_i\}_i$ will be a basis of H , and $\{e^i\}_i$ the corresponding dual basis of H^* . $\Omega = \Omega^1 \otimes \cdots \otimes \Omega^5 \in H^{\otimes 5}$ is defined by

$$\Omega = X_{(1,1)}^1 y^1 x^1 \otimes X_{(1,2)}^1 y^2 x_1^2 \otimes X_2^1 y^3 x_2^2 \otimes S^{-1}(f^1 X^2 x^3) \otimes S^{-1}(f^2 X^3), \quad (5.1)$$

where $f = f^1 \otimes f^2$ is the twist of Drinfeld defined in (2.11). As a vector space, $D(H) = H^* \otimes H$. The multiplication is given by the formula

$$(\varphi \bowtie h)(\psi \bowtie h') = [(\Omega^1 \rightarrow \varphi \leftarrow \Omega^5)(\Omega^2 h_{(1,1)} \rightarrow \psi \leftarrow S^{-1}(h_2)\Omega^4)] \bowtie \Omega^3 h_{(1,2)} h', \quad (5.2)$$

for all $\varphi, \psi \in H^*$ and $h, h' \in H$, where we wrote $\varphi \bowtie h$ in place of $\varphi \otimes h$ in order to distinguish this new multiplication on $H^* \otimes H$. The unit of $D(H)$ is $\varepsilon \bowtie 1$. The explicit formulas for the comultiplication, counit, reassociator and antipode are

$$\begin{aligned}
\Delta_D(\varphi \bowtie h) &= (\varepsilon \bowtie X^1 Y^1)(p_1^1 x^1 \rightarrow \varphi_2 \leftarrow Y^2 S^{-1}(p^2) \bowtie p_2^1 x^2 h_1) \\
&\quad \otimes (X_1^2 \rightarrow \varphi_1 \leftarrow S^{-1}(X^3) \bowtie X_2^2 Y^3 x^3 h_2),
\end{aligned} \quad (5.3)$$

$$\varepsilon_D(\varphi \bowtie h) = \varepsilon(h)\varphi(S^{-1}(\alpha)), \quad (5.4)$$

$$\Phi_D = (\varepsilon \bowtie X^1) \otimes (\varepsilon \bowtie X^2) \otimes (\varepsilon \bowtie X^3), \quad (5.5)$$

$$S_D(\varphi \bowtie h) = (\varepsilon \bowtie S(h)f^1)(p_1^1 U^1 \rightharpoonup \bar{S}^{-1}(\varphi) \leftarrow f^2 S^{-1}(p^2) \bowtie p_2^1 U^2), \quad (5.6)$$

and

$$\alpha_D = \varepsilon \bowtie \alpha; \quad \beta_D = \varepsilon \bowtie \beta. \quad (5.7)$$

Here $p_R = p^1 \otimes p^2$, $f = f^1 \otimes f^2$ and $U = U^1 \otimes U^2$ are the elements defined by (2.17), (2.11) and (3.5). $D(H)$ is a quasi-Hopf algebra and H is a quasi-Hopf subalgebra via the canonical morphism $i_D: H \rightarrow D(H)$, $i_D(h) = \varepsilon \bowtie h$.

Now we take a non-zero left cointegral λ on H and a non-zero right integral r in H , and define $T := \mu^{-1}(\delta^2)\delta^1 \rightharpoonup \lambda \in H^*$, where μ is the modular element of H^* and $\delta = \delta^1 \otimes \delta^2$ is the element of $H \otimes H$ defined in (2.10). We claim that $T \bowtie r$ is a non-zero left and right integral in $D(H)$. That it is non-zero follows easily from the fact that r is non-zero and $T(r) = \mu^{-1}(\delta^2)\lambda(r\delta^1) = \mu^{-1}(\varepsilon(\delta^1)\delta^2)\lambda(r) = \mu^{-1}(\beta)\lambda(r) \neq 0$, as $\mathcal{L} \times \int_r^H \ni (\lambda', r') \mapsto \lambda'(r') \in k$ is non-degenerated, see [11, Lemma 4.4].

The difficult part is to show that $T \bowtie r$ is a left and right integral in $D(H)$. To prove that it is a left integral we need the following formulas:

$$\Delta(h_1)\delta(S \otimes S)(\Delta^{\text{cop}}(h_2)) = \varepsilon(h)\delta, \quad \text{for all } h \in H, \quad (5.8)$$

$$Y^1 \delta^1 S(Y_2^3) \otimes Y^2 \delta^2 S(Y_1^3) = \beta S(\tilde{p}^2) \otimes S(\tilde{p}^1), \quad (5.9)$$

$$z^1 \tilde{p}^1 \otimes z^2 \tilde{p}_1^2 \otimes z^3 \tilde{p}_2^2 = Y_1^2 Z^2 S^{-1}(Y^1 Z^1 \beta) \otimes Y_2^2 Z^3 \otimes Y^3, \quad (5.10)$$

$$X^1 \otimes q^1 X_1^2 \otimes S^{-1}(X^3) q^2 X_2^2 = q_1^1 x^1 \otimes q_2^1 x^2 \otimes q^2 x^3. \quad (5.11)$$

Formula (5.8) can be found in [6], and (5.9)–(5.11) can be deduced easily from the definitions of δ , p_L and q_R , and the axioms (2.3) and (2.5). We leave the details to the reader.

Proposition 5.1. *Keeping the above setting and notation, we have that $T \bowtie r$ is a left integral in $D(H)$.*

Proof. We check this assertion by direct computation. If $\varphi \in H^*$ and $h \in H$ then

$$\begin{aligned} (\varphi \bowtie h)(T \bowtie r) &= \mu^{-1}(\delta^2)(\Omega^1 \rightharpoonup \varphi \leftarrow \Omega^5)(\Omega^2 h_{(1,1)} \delta^1 \rightharpoonup \lambda \leftarrow S^{-1}(h_2) \Omega^4) \bowtie \Omega^3 h_{(1,2)} r \\ &\stackrel{(3.9)}{=} \mu^{-1}(\Omega^3 h_{(1,2)} \delta^2)(\Omega^1 \rightharpoonup \varphi \leftarrow \Omega^5)(\Omega^2 h_{(1,1)} \delta^1 \rightharpoonup \lambda \leftarrow S^{-1}(h_2) \Omega^4) \bowtie r \\ &\stackrel{(3.20)}{=} \mu^{-1}(\Omega^3 h_{(1,2)} \delta^2 S((h_2)_1))(\Omega^2 h_{(1,1)} \delta^1 S((h_2)_2) \rightharpoonup \lambda \leftarrow \Omega^4) \bowtie \Omega^3 h_{(1,2)} r \\ &\stackrel{(5.8)}{=} \varepsilon(h) \mu^{-1}(\Omega^3 \delta^2)(\Omega^1 \rightharpoonup \varphi \leftarrow \Omega^5)(\Omega^2 \delta^1 \rightharpoonup \lambda \leftarrow \Omega^4) \bowtie r. \end{aligned}$$

Therefore it suffices to show that

$$\mu^{-1}(\Omega^3 \delta^2) \lambda(\Omega^4 h_2 \Omega^2 \delta^1) \Omega^5 h_1 \Omega^1 = \mu^{-1}(\delta^2) \lambda(h \delta^1) S^{-1}(\alpha),$$

for all $h \in H$. To this end, we compute

$$\begin{aligned} &\mu^{-1}(\Omega^3 \delta^2) \lambda(\Omega^4 h_2 \Omega^2 \delta^1) \Omega^5 h_1 \Omega^1 \\ &\stackrel{(5.1)}{=} \mu^{-1}(X_2^1 y^3 x_2^2 \delta^2) \lambda(S^{-1}(f^1 X^2 x^3) h_2 X_{(1,2)}^1 y^2 x_1^2 \delta^1) S^{-1}(f^2 X^3) h_1 X_{(1,1)}^1 y^1 x^1 \end{aligned}$$

$$\begin{aligned}
& \stackrel{(2.3), (5.9)}{=} \mu^{-1}(X_2^1 y^2 x_1^3 S(\tilde{p}^1)) \lambda(S^{-1}(f^1 X^2 y^3 x_2^3) h_2 X_{(1,2)}^1 y_2^1 x^2 \beta S(\tilde{p}^2)) \\
& \quad \times S^{-1}(f^2 X^3) h_1 X_{(1,1)}^1 y_1^1 x^1 \\
& \stackrel{(3.20), (2.22)}{=} \mu^{-1}(X_2^1 y^2 S(\tilde{p}^1)) \lambda(S^{-1}(f^1 X^2 y^3) h_2 X_{(1,2)}^1 y_2^1 x^2 \beta S(\tilde{p}^2 x^3)) \\
& \quad \times S^{-1}(f^2 X^3) h_1 X_{(1,1)}^1 y_1^1 x^1 \\
& \stackrel{(2.17), (3.15)}{=} \mu^{-1}(X_2^1 y^2 S(\tilde{p}^1)) \lambda(S^{-1}(f^1 X^2 y^3) (h X_1^1 y^1 S(\tilde{p}^1))_2 U^2 S(\tilde{p}^2)) \\
& \quad \times S^{-1}(f^2 X^3) (h X_1^1 y^1 S(\tilde{p}^1))_1 U^1 \tilde{p}^2 \\
& \stackrel{(3.20), (3.12)}{=} \mu^{-1}(X_2^1 y^2 S(\tilde{p}^1)) \mu(X_1^2 y_1^3) \langle \lambda, S^{-1}(f^1) (h X_1^1 y^1 S(X_{(2,1)}^2 y_{(2,1)}^3 \tilde{p}_1^2 \tilde{p}^1))_2 U^2 \rangle \\
& \quad \times S^{-1}(f^2 X^3) (h X_1^1 y^1 S(X_{(2,1)}^2 y_{(2,1)}^3 \tilde{p}_1^2 \tilde{p}^1))_1 U^1 X_{(2,2)}^2 y_{(2,2)}^3 \tilde{p}_2^2 \tilde{p}^2 \\
& \stackrel{(3.7)}{=} \mu^{-1}(X_2^1 y^2 S(\tilde{p}^1)) \mu(X_1^2 y_1^3) \mu(q_1^1 z^1) \langle \lambda, h X_1^1 y^1 S(q_2^1 z^2 X_{(2,1)}^2 y_{(2,1)}^3 \tilde{p}_1^2 \tilde{p}^1) \rangle \\
& \quad \times S^{-1}(X^3) q^2 z^3 X_{(2,2)}^2 y_{(2,2)}^3 \tilde{p}_2^2 \tilde{p}^2 \\
& \stackrel{(2.1)}{=} \mu^{-1}(X_2^1 y^2) \mu(q_1^1 X_{(1,1)}^2 y_{(1,1)}^3 z^1 \tilde{p}^1) \langle \lambda, h X_1^1 y^1 S(q_2^1 X_{(1,2)}^2 y_{(1,2)}^3 z^2 \tilde{p}_1^2 \tilde{p}^1) \rangle \\
& \quad \times S^{-1}(X^3) q^2 X_2^2 y_2^3 z^3 \tilde{p}_2^2 \tilde{p}^2 \\
& \stackrel{(5.10), (5.11)}{=} \mu((q_2^1)_1 (x^2 y_1^3 Y^2)_1 Z^2 S^{-1}(q_{(1,2)}^1 x_2^1 y^2 Y^1 Z^1 \beta)) \\
& \quad \times \langle \lambda, h q_{(1,1)}^1 x_1^1 y^1 S((q_2^1)_2 (x^2 y_1^3 Y^2)_2 Z^3 \tilde{p}^1) \rangle q^2 x^3 y_2^3 Y^3 \tilde{p}^2 \\
& \stackrel{(2.3), (2.1), (2.5)}{=} \mu((q_2^1 z^3)_1 Z^2 S^{-1}(q_{(1,2)}^1 z^2 Z^1 \beta)) \langle \lambda, h q_{(1,1)}^1 z^1 y^1 S((q_2^1 z^3)_2 Z^3 y^2 \tilde{p}^1) \rangle q^2 y^3 \tilde{p}^2 \\
& \stackrel{(2.1)}{=} \mu^{-1}(z^2 (q_2^1)_1 Z^1 \beta S(z_1^3 (q_2^1)_{(2,1)} Z^2)) \langle \lambda, h z^1 q_1^1 y^1 S(z_2^3 (q_2^1)_{(2,2)} Z^3 y^2 \tilde{p}^1) \rangle q^2 y^3 \tilde{p}^2 \\
& \stackrel{(2.1), (2.5)}{=} \mu^{-1}(z^2 Z^1 \beta S(z_1^3 Z^2)) \langle \lambda, h z^1 q_1^1 y^1 S(z_2^3 Z^3 q_2^1 y^2 \tilde{p}^1) \rangle q^2 y^3 \tilde{p}^2 \\
& \stackrel{(2.19), (2.5)}{=} \mu^{-1}(z^2 Z^1 \beta S(z_1^3 Z^2)) \langle \lambda, h z^1 \beta S(z_2^3 Z^3) \rangle S^{-1}(\alpha) \\
& \stackrel{(2.10)}{=} \mu^{-1}(\delta^2) \lambda(h \delta^1) S^{-1}(\alpha),
\end{aligned}$$

for all $h \in H$, and this finishes the proof. \square

Corollary 5.2. *The quantum double $D(H)$ is a semisimple algebra if and only if H is semisimple and admits a normalized left cointegral, that is a left cointegral λ satisfying $\lambda(S^{-1}(\alpha)\beta) \neq 0$.*

Proof. This is an immediate consequence of the Maschke theorem for quasi-Hopf algebras proved in [16]. Note that, for the non-zero left integral $\mathbb{T} = \mu^{-1}(\delta^2)\delta^1 \rightharpoonup \lambda \bowtie r$ in $D(H)$ we have $\varepsilon_D(\mathbb{T}) = \varepsilon(r)\mu^{-1}(\delta^2)\lambda(S^{-1}(\alpha)\delta^1)$, and so $\varepsilon_D(\mathbb{T}) \neq 0$ if and only if $\varepsilon(r) \neq 0$ and $\mu^{-1}(\delta^2)\lambda(S^{-1}(\alpha)\delta^1) \neq 0$. But $\varepsilon(r) \neq 0$ implies H semisimple, and therefore unimodular, in which case $\mu^{-1}(\delta^2)\delta^1 = \beta$. \square

Examples 5.3. (1) $D(H(2))$ is semisimple because $H(2)$ is semisimple and the left cointegral P_g on $H(2)$ found in Example 3.8 satisfies $P_g(S^{-1}(g)) = P_g(g) = 1$.

(2) $D(H_{\pm}(8))$ is not semisimple because $H_{\pm}(8)$ is not semisimple, cf. Example 3.4.

From [4, Theorem 6.5] we know that $\mathbb{T} = T \bowtie r$ is a right integral in $D(H)$. In Appendix A, we will present a direct proof, and we will also show that

$$\begin{aligned}
& \lambda(S^{-1}(f^2)h_1g^1S(h'))S^{-1}(f^1)h_2g^2 \\
&= \mu(f^1)\mu^{-1}(U_2^2\mathbf{U}^2\alpha)\mu(\beta) \\
&\quad \times \mu(U^1y_2^1x^2)\lambda(hS(y^3x_2^3h_2'\tilde{p}^2))S^{-1}(\underline{g}^{-1}y_1^1x^1)S(S(U_1^2\mathbf{U}^1y^2x_1^3h_1'\tilde{p}^1)f^2), \quad (5.12)
\end{aligned}$$

for all $\lambda \in \mathcal{L}$ and $h, h' \in H$, where $U = U^1 \otimes U^2 = \mathbf{U}^1 \otimes \mathbf{U}^2$ is the element defined in (3.5). The proof can be found right after the end of the proof of Lemma A.1.

Now we will focus on the cointegrals on $D(H)$. In the sequel, we will identify $D(H)^* \cong H \otimes H^*$.

Proposition 5.4. Take non-zero elements $\lambda \in \mathcal{L}$ and $r \in \int_r^H$. Then

$$\Gamma = r \bowtie \mu(\tilde{p}^1)S(\tilde{p}^2) \rightarrow \lambda \leftarrow \mu^{-1}(f^1)S^{-1}(f^2) \in D(H)^* \quad (5.13)$$

is a non-zero left cointegral on $D(H)$.

Proof. It is clear that $\Gamma \neq 0$. Since H is a quasi-Hopf subalgebra of $D(H)$ via i_D it follows that the elements U and V for $D(H)$ are $U_D = \varepsilon \bowtie U^1 \otimes \varepsilon \bowtie U^2$ and $V_D = \varepsilon \bowtie V^1 \otimes \varepsilon \bowtie V^2$. Applying (5.11) to H^{op} , we have

$$X^1p_1^1 \otimes X^2p_2^1 \otimes X^3p^2 = x^1 \otimes x_1^2p^1 \otimes x_2^2p^2S(x^3). \quad (5.14)$$

Identifying $D(H)^* \cong H \otimes H^*$, we compute

$$\begin{aligned}
& \Gamma((\varepsilon \bowtie V^2)(X_1^2 \rightarrow \varphi_1 \leftarrow S^{-1}(X^3) \bowtie X_2^2Y^3x^3h_2)(\varepsilon \bowtie U^2)) \\
&\quad \times (\varepsilon \bowtie V^1)(\varepsilon \bowtie X^1Y^1)(p_1^1x^1 \rightarrow \varphi_2 \leftarrow Y^2S^{-1}(p^2) \bowtie p_2^1x^2h_1)(\varepsilon \bowtie U^1) \\
&\stackrel{(5.2)}{=} \varphi_1(S^{-1}(V_2^2X^3)rV_{(1,1)}^2X_1^2)\lambda(S^{-1}(f^2)V_{(1,2)}^2X_2^2Y^3x^3h_2U^2S(\tilde{p}^2))\mu(\tilde{p}^1) \\
&\quad \times \mu^{-1}(f^1)(\varepsilon \bowtie V^1X^1Y^1)(p_1^1x^1 \rightarrow \varphi_2 \leftarrow Y^2S^{-1}(p^2) \bowtie p_2^1x^2h_1U^1) \\
&= \varphi(r)\mu(S^{-1}(f^1)V_2^2X^3)\mu(\tilde{p}^1)\lambda(S^{-1}(f^2)V_1^2X^2h_2U^2S(\tilde{p}^2))\varepsilon \bowtie V^1X^1h_1U^1 \\
&\stackrel{(3.5), (2.8)}{=} \varphi(r)\mu^{-1}(S(X^3)F^1f_1^1p_1^1)\mu(\tilde{p}^1)\lambda(S^{-1}(S(X^2)F^2f_2^1p_2^1)h_2U^2S(\tilde{p}^2)) \\
&\quad \times \varepsilon \bowtie S^{-1}(S(X^1)f^2p^2)h_1U^1 \\
&\stackrel{(2.14), (5.14)}{=} \varphi(r)\mu^{-1}(f^1x^1)\mu(\tilde{p}^1)\lambda(S^{-1}(F^1f_1^2x_1^2p^1)h_2U^2S(\tilde{p}^2)) \\
&\quad \times \varepsilon \bowtie S^{-1}(F^2f_2^2x_2^2p^2S(x^3))h_1U^1 \\
&\stackrel{(2.8), (3.5), (3.12)}{=} \varphi(r)\mu^{-1}(f^1x^1)\mu(\tilde{p}^1)\lambda(V^2(S^{-1}(f^2x^2)hS(\tilde{p}_1^2))_2U^2) \\
&\quad \times \varepsilon \bowtie x^3V^1(S^{-1}(f^2x^2)hS(\tilde{p}_1^2))_1U^1\tilde{p}_2^2 \\
&\stackrel{(3.6), (3.20)}{=} \varphi(r)\mu^{-1}(f^1x^1)\mu(x_1^2y^1\tilde{p}^1)\lambda(S^{-1}(f^2)hS(x_2^2y^2\tilde{p}_1^2))\varepsilon \bowtie x^3y^3\tilde{p}_2^2 \\
&\stackrel{(2.19), (2.3)}{=} \varphi(r)\mu(\tilde{p}^1)\mu^{-1}(f^1)\lambda(S^{-1}(f^2)hS(\tilde{p}^2))\varepsilon \bowtie 1 = \Gamma(\varphi \bowtie h)\varepsilon \bowtie 1.
\end{aligned}$$

As $D(H)$ is unimodular the above computation shows that Γ is a left cointegral on $D(H)$, as desired. \square

Corollary 5.5. $D(H)$ admits a normalized left cointegral if and only if $D(H)$ is a semisimple algebra.

Proof. For the non-zero left cointegral Γ defined in (5.13) we have

$$\Gamma(S_D^{-1}(\varepsilon \bowtie \alpha)(\varepsilon \bowtie \beta)) = \Gamma(\varepsilon \bowtie S^{-1}(\alpha)\beta) = \varepsilon(r)\mu(\tilde{p}^1)\mu^{-1}(f^1)\lambda(S^{-1}(\alpha f^2)\beta \tilde{p}^2).$$

This scalar is non-zero if and only if $\varepsilon(r) \neq 0$ and $\mu(\tilde{p}^1)\mu^{-1}(f^1)\lambda(S^{-1}(\alpha f^2)\beta \tilde{p}^2) \neq 0$. But, as we have already mentioned, $\varepsilon(r) \neq 0$ implies H unimodular, and in this case $\mu(\tilde{p}^1)\mu^{-1}(f^1)\lambda(S^{-1}(\alpha f^2)\beta \tilde{p}^2) = \lambda(S^{-1}(\alpha)\beta)$. Then the result follows from Corollary 5.2. \square

Now we describe the space of right cointegrals on $D(H)$.

Proposition 5.6. *If $t \in \int_l^H$ and $\lambda \in \mathcal{L}$ are non-zero then $t \bowtie \lambda \circ S$ is a non-zero right cointegral on $D(H)$.*

Proof. Since $D(H)$ is unimodular $\Gamma \circ S_D$ is a non-zero right cointegral on $D(H)$, cf. Corollary 4.10. So it suffices to show that $\Gamma \circ S_D = S(r) \bowtie \lambda \circ S$. Applying μ to the both sides of (5.12) we obtain after a straightforward computation that

$$\mu(S^{-1}(f^1)h_2g^2)\lambda(S^{-1}(f^2)h_1g^1S(h')) = \mu^{-1}(\alpha \underline{g}^{-1})\mu(\tilde{q}^1)\mu(\tilde{q}_1^2h'_1\tilde{p}^1)\lambda(hS(\tilde{q}_2^2h'_2\tilde{p}^2)),$$

for all $h, h' \in H$. Consequently, by (2.27) we obtain that

$$\begin{aligned} & \mu^{-1}(\tilde{q}^1)\mu(S^{-1}(f^1)S(h)_2g^2)\lambda(S^{-1}(f^1)S(h)_1g^1S(h'\tilde{q}^2)) \\ &= \mu^{-1}(\alpha \underline{g}^{-1})\mu(\tilde{q}^1)\mu(\tilde{q}_1^2h'_1)\lambda(S(\tilde{q}_2^2h'_2h)), \end{aligned} \quad (5.15)$$

for all $h, h' \in H$. We then compute

$$\begin{aligned} \Gamma \circ S_D(\varphi \bowtie h) &= \Gamma((\varepsilon \bowtie S(h)f^1)(p_1^1U^1 \rightharpoonup \varphi \circ S^{-1} \leftarrow f^2S^{-1}(p^2) \bowtie p_2^1U^2)) \\ &= \varphi \circ S^{-1}(f^2S^{-1}(S(h)_2f_2^1p^2)rS(h)_{(1,1)}f_{(1,1)}^1p_1^1U^1) \\ &\quad \times \mu(\tilde{p}^1)\mu^{-1}(F^1)\lambda(S^{-1}(F^2)S(h)_{(1,2)}f_{(1,2)}^1p_2^1U^2S(\tilde{p}^2)) \\ &\stackrel{(4.14)}{=} \varphi(S^{-1}(r))\mu(S^{-1}(F^1)S(h)_2g^2S(\tilde{q}^1))\mu(\tilde{p}^1)\lambda(S^{-1}(F^2)S(h)_1g^1S(\tilde{p}^2\tilde{q}^2)) \\ &\stackrel{(5.15)}{=} \mu^{-1}(\alpha \underline{g}^{-1})\varphi(S^{-1}(r))\mu((\tilde{q}^2\tilde{p}^2)_1)\mu(\tilde{q}^1\tilde{p}^1)\lambda(S((\tilde{q}^2\tilde{p}^2)_2h)) \\ &\stackrel{(2.18), (2.19)}{=} \mu^{-1}(\underline{g}^{-1})\varphi(S^{-1}(r))\mu^{-1}(\alpha)\mu(\alpha)\mu^{-1}(\beta)\lambda(S(h)) \\ &\stackrel{(3.10)}{=} \varphi((\mu^{-1}(\underline{g})\mu(\beta))^{-1}S^{-1}(r))\lambda(S(h)) = (S(r) \bowtie \lambda \circ S)(\varphi \bowtie h), \end{aligned}$$

for all $\varphi \in H^*$ and $h \in H$, where in the last equality we used Corollary 4.13. \square

The modular element of $D(H)^*$ is $\mu_D = \varepsilon_D$. Our next aim is to compute the modular element \underline{g}_D of $D(H)$. To this end, we will need an explicit formula for the inverse of the antipode S_D of $D(H)$ and a lemma.

Lemma 5.7. *The composition inverse S_D^{-1} of the antipode of $D(H)$ is given by the formula*

$$S_D^{-1}(\varphi \bowtie h) = (\varepsilon \bowtie S^{-1}(f^2h))(p_1^1S^{-1}(q^2g^2) \rightharpoonup \varphi \circ S \leftarrow S^{-1}(p^2f^1) \bowtie p_2^1S^{-1}(q^1g^1)),$$

for all $\varphi \in H^*$ and $h \in H$.

Proof. We first observe that (2.20) and (2.25) imply that

$$(\varepsilon \bowtie q^1 h_1)(p_1^1 \rightarrow \varphi \leftarrow q^2 h_2 S^{-1}(p^2) \bowtie p_2^1) = h_1 \rightarrow \varphi \bowtie h_2,$$

for all $\varphi \in H^*$ and $h \in H$. Consequently,

$$\begin{aligned} & (\varepsilon \bowtie q^1 S(P^1)_1)(p_1^1 U^1 P^2 f^1 \rightarrow \varphi \leftarrow q^2 S(P^1)_2 S^{-1}(p^2) \bowtie p_2^1 U^2)(\varepsilon \bowtie f^2) \\ &= (S(P^1)_1 U^1 P^2 f^1 \rightarrow \varphi \bowtie S(P^1)_2 U^2)(\varepsilon \bowtie f^2) \\ &\stackrel{(3.5), (2.8)}{=} (g^1 S(q^2 P_2^1) P^2 f^1 \rightarrow \varphi \bowtie g^2 S(q^1 p_1^1))(\varepsilon \bowtie f^2) \stackrel{(2.24)}{=} \varphi \bowtie 1, \end{aligned}$$

for all $\varphi \in H^*$. By the definition of S_D we have

$$S_D(S^{-1}(g^2) \rightarrow \varphi \bowtie S^{-1}(g^1)) = p_1^1 U^1 \rightarrow \varphi \circ S^{-1} \leftarrow S^{-1}(p^2) \bowtie p_2^1 U^2,$$

and combining these two relations we find for all $\varphi \in H^*$ that

$$(\varepsilon \bowtie q^1 S(P^1)_1) S_D(S^{-1}(g^2) \rightarrow (P^2 f^1 \rightarrow \varphi \leftarrow q^2 S(P^1)_2) \circ S \bowtie S^{-1}(g^1))(\varepsilon \bowtie f^2)$$

equals $\varphi \bowtie 1$. As H is a quasi-Hopf subalgebra of $D(H)$ it follows that $S_D^{-1}(\varepsilon \bowtie h) = \varepsilon \bowtie S^{-1}(h)$, for all $h \in H$. This and the fact that S_D^{-1} is an anti-algebra morphism lead us to the equality

$$\begin{aligned} S_D^{-1}(\varphi \bowtie h) &= S_D^{-1}(\varepsilon \bowtie h) S_D^{-1}(\varphi \bowtie 1) \\ &= (\varepsilon \bowtie S^{-1}(f^2 h)) (S^{-1}(q^2 S(P^1)_2 g^2) \rightarrow \varphi \circ S \leftarrow S^{-1}(P^2 f^1) \bowtie S^{-1}(q^1 S(P^1)_1 g^1)) \\ &\stackrel{(2.8)}{=} (\varepsilon \bowtie S^{-1}(f^2 h)) (P_1^1 S^{-1}(q^2 g^2) \rightarrow \varphi \circ S \leftarrow S^{-1}(P^2 f^1) \bowtie P_2^1 S^{-1}(q^1 g^1)), \end{aligned}$$

for all $\varphi \in H^*$ and $h \in H$. \square

Lemma 5.8. Let H be a finite dimensional quasi-Hopf algebra. Then for all $r \in \int_r^H$ and $h \in H$ the following equalities hold:

$$hr_1 \otimes r_2 = \mu^{-1}(h_1 p^1) q^1 r_1 \otimes S^{-1}(h_2 p^2) q^2 r_2, \quad (5.16)$$

$$r_1 U^1 \otimes r_2 U^2 S(h) = r_1 U^1 h \otimes r_2 U^2, \quad (5.17)$$

$$V^1 r_1 \otimes S^{-1}(h) V^2 r_2 = \mu(h_1) h_2 V^1 r_1 \otimes V^2 r_2. \quad (5.18)$$

Proof. (5.16) follows since

$$\begin{aligned} hr_1 \otimes r_2 &\stackrel{(2.24)}{=} h q^1 p_1^1 r_1 \otimes S^{-1}(p^2) q^2 p_2^1 r_2 \\ &\stackrel{(2.21)}{=} q^1 (h_1 p^1 r)_1 \otimes S^{-1}(h_2 p^2) q^2 (h_1 p^1 r)_2 \\ &= \mu^{-1}(h_1 p^1) q^1 r_1 \otimes S^{-1}(h_2 p^2) q^2 r_2. \end{aligned}$$

(5.17) is a direct consequence of (3.12) while (5.18) can be proved with the help of (3.13). We leave the verifications to the reader. \square

In order to compute the modular element \underline{g}_D of $D(H)$ we need a left integral \mathbb{T} in $D(H)$ and a left cointegral Γ on $D(H)$ such that $\Gamma(S_D^{-1}(\mathbb{T})) = 1$. Since $\mu_D = \varepsilon_D$ it turns out that this is equivalent to $\Gamma(\mathbb{T}) = 1$. Also note that the unimodularity of $D(H)$ implies that $\Gamma \circ S_D = \Gamma \circ S_D^{-1}$.

Now take $\mathbb{T} = \mu^{-1}(\delta^2) \rightarrow \lambda' \bowtie r'$ for some $0 \neq \lambda' \in \mathcal{L}$ and $0 \neq r' \in \int_r^H$, and let Γ be defined as in (5.13). A simple inspection ensures that

$$\Gamma \circ S_D^{-1}(\mathbb{T}) = \mu(\delta^1)\mu^{-1}(\delta^2)\lambda'(S(r))\lambda(S(r'))$$

and since $S(\delta^1)\alpha\delta^2 = S(\beta)$ and $\varepsilon(\underline{g}) = \mu(\beta)$, by Remarks 4.2 (i) we conclude that $\Gamma \circ S_D^{-1}(\mathbb{T}) = \mu^{-1}(\alpha)^{-1}\lambda'(S(r))\lambda(r')$. Thus we have to consider λ, λ' and r, r' such that $\lambda'(S(r))\lambda(r') = \mu^{-1}(\alpha)$.

Proposition 5.9. *The modular element \underline{g}_D of $D(H)$ is given by*

$$\begin{aligned}\underline{g}_D &= \mu(g_1^1)\mu^{-1}(g^2)S_D^{-1}(\mu \bowtie g_2^1 S^{-2}(\underline{g}^{-1})) \\ &= \mu(\tilde{q}^1 g^1)\mu^{-1}(\tilde{p}^1)(\varepsilon \bowtie S^{-3}(\underline{g}^{-1}))(\mu^{-1} \bowtie (S^{-1}(\tilde{q}^2 g^2) \leftarrow \mu^{-1})\tilde{p}^2).\end{aligned}$$

Proof. Let $\lambda, \lambda' \in \mathcal{L}$ and $r, r' \in \int_r^H$ be such that $\lambda'(S(r))\lambda(r') = \mu^{-1}(\alpha)$. Then, by the above comments, (4.4) and (5.3), the modular element \underline{g}_D can be computed as follows:

$$\begin{aligned}\underline{g}_D &= \Gamma \circ S_D^{-1}((q_1^2 X^2)_1 \rightarrow (\delta^1 \rightarrow \lambda')_1 \leftarrow S^{-1}(X^3) \bowtie (q_1^2 X^2)_2 Y^3 x^3 r'_2 P^2) \\ &\quad \times \mu^{-1}(\delta^2)S_D^{-1}((\varepsilon \bowtie q^1 X_1^1 Y^1)(p_1^1 x^1 \rightarrow (\delta^1 \rightarrow \lambda')_2 \leftarrow Y^2 S^{-1}(p^2) \bowtie p_2^1 x^2 r'_1 P^1)) \\ &= \mu(q_1^2)\lambda(S(q_2^2 Y^3 x^3 r'_2 P^2))\lambda'(S(r))\mu(Y^2 S^{-1}(p^2))\mu(p_1^1 x^1) \\ &\quad \times \mu^{-1}(\delta^2)\mu(\delta^1)S_D^{-1}((\varepsilon \bowtie q^1 Y^1)(\mu \bowtie p_2^1 x^2 r'_1 P^1)) \\ &\stackrel{(5.16)}{=} \mu^{-1}(p_{(2,1)}^1 x_1^1 \mathbb{P}^1)\lambda'(S(r))\mu(q_1^2)\lambda(S(q_2^2 Y^3 x^3 S^{-1}(p_{(2,2)}^1 x_2^2 \mathbb{P}^2) Q^2 r'_2 P^2)) \\ &\quad \times \mu^{-1}(\alpha)^{-1}\mu(\beta)\mu(Y^2 S^{-1}(p^2))\mu(p_1^1 x^1)S_D^{-1}((\varepsilon \bowtie q^1 Y^1)(\mu \bowtie Q^1 r'_1 P^1)) \\ &\stackrel{(5.14), (2.1)}{=} \mu^{-1}(X^2 S^{-1}(X^1 \beta))\mu^{-1}(\alpha)^{-1}\lambda'(S(r))\mu(Y^2 S^{-1}(p^2))\mu(q_1^2) \\ &\quad \times \lambda(S(q_2^2 Y^3 S^{-1}(X^3 p^1 \beta) Q^2 r'_2 P^2))S_D^{-1}((\varepsilon \bowtie q^1 Y^1)(\mu \bowtie Q^1 r'_1 P^1)) \\ &\stackrel{(3.9), (4.11)}{=} \mu^{-1}(\alpha)^{-1}\lambda'(S(r))\mu(q_1^2)\mu(Y^2 S^{-1}(p^2))\lambda(S(q_2^2 Y^3 S^{-1}(p^1 \beta) V^2 r'_2 U^2)) \\ &\quad \times S_D^{-1}((\varepsilon \bowtie q^1 Y^1)(\mu \bowtie V^1 r'_1 U^1)) \\ &\stackrel{(5.18)}{=} \mu^{-1}(q_2^1 Y_2^1 p^2 S(Y^2))\lambda'(S(r))\mu(q_1^2)\lambda(S(q_2^2 Y^3 S^{-1}(q_1^1 Y_1^1 p^1 \beta) V^2 r'_2 U^2)) \\ &\quad \times \mu^{-1}(\alpha)^{-1}S_D^{-1}(\mu \bowtie V^1 r'_1 U^1) \\ &\stackrel{(2.10)}{=} \mu^{-1}(q_2^1 \delta^2 S(q_1^2))\lambda'(S(r))\lambda \circ S(S^{-1}(q_1^1 \delta^1 S(q_2^2)) V^2 r'_2 U^2) \\ &\quad \times \mu^{-1}(\alpha)^{-1}S_D^{-1}(\mu \bowtie V^1 r'_1 U^1) \\ &\stackrel{(2.13), (2.8)}{=} \mu^{-1}((q^1 \beta S(q^2))_2 g^2)\lambda'(S(r))\lambda \circ S(S^{-1}((q^1 \beta S(q^2))_1 g^1) V^2 r'_2 U^2) \\ &\quad \times \mu^{-1}(\alpha)^{-1}S_D^{-1}(\mu \bowtie V^1 r'_1 U^1)\end{aligned}$$

$$\begin{aligned}
& \stackrel{(2.5), (5.18)}{=} \mu(g_1^1) \mu^{-1}(g^2) \mu^{-1}(\alpha)^{-1} \lambda'(S(r)) \lambda \circ S(V^2 r'_2 U^2) S_D^{-1}(\mu \bowtie g_2^1 V^1 r'_1 U^1) \\
& \stackrel{(3.6)}{=} \mu(g_1^1) \mu^{-1}(g^2) S_D^{-1}(\mu \bowtie g_2^1 S^{-2}(\underline{g}^{-1})).
\end{aligned}$$

In the second equality we used Proposition 5.6 and the fact that $\Gamma \circ S_D^{-1} = \Gamma \circ S_D$, in the third one, we used the properties $S(r) \in \int_1^H$ and μ is an algebra map, and in the last equality Remarks 4.2 (i) and (5.17). We have also denoted by $\mathbb{P}^1 \otimes \mathbb{P}^2$ another copy of p_R . This proves the first formula for \underline{g}_D . For the second one we use the form of S_D^{-1} found above to compute

$$\begin{aligned}
\underline{g}_D &= \mu(g_1^1) \mu^{-1}(g^2) (\varepsilon \bowtie S^{-3}(\underline{g}^{-1}) S^{-1}(f^2 g_2^1)) \\
&\quad \times (p_1^1 S^{-1}(q^2 G^2) \rightharpoonup \mu^{-1} \leftarrow S^{-1}(p^2 f^1) \bowtie p_2^1 S^{-1}(q^1 G^1)) \\
&\stackrel{(5.2)}{=} \mu^{-1}((S^{-1}(f^2 g_2^1)_1 p^1)_1 S^{-1}(q^2 G^2)) \mu(S^{-1}(f^2 g_2^1)_2 p^2 f^1) \mu(g_1^1) \\
&\quad \times \mu^{-1}(g^2) (\varepsilon \bowtie S^{-3}(\underline{g}^{-1})) (\mu^{-1} \bowtie (S^{-1}(f^2 g_2^1)_1 p^1)_2 S^{-1}(q^1 G^1)) \\
&\stackrel{(2.8), (2.14)}{=} \mu^{-1}(S^{-1}(f^2 x^3 g_{(2,2)}^1 \mathbb{G}^2)_1 S^{-1}(q^2 G^2)) \mu^{-1}(g^2) \\
&\quad \times \mu(S^{-1}(F^2 f_2^1 x^2 g_{(2,1)}^1 \mathbb{G}^1) \beta F^1 f_1^1 x^1 g_1^1) \\
&\quad \times (\varepsilon \bowtie S^{-3}(\underline{g}^{-1})) (\mu^{-1} \bowtie S^{-1}(f^2 x^3 g_{(2,2)}^1 \mathbb{G}^2)_2 S^{-1}(q^1 G^1)) \\
&\stackrel{(2.15), (2.1), (2.5)}{=} \mu^{-1}(S^{-1}(\alpha x^2 \mathbb{G}^1) x^1) \mu^{-1}(S^{-1}(g^1 x^3 \mathbb{G}^2)_1 S^{-1}(q^2 G^2)) \mu^{-1}(g^2) \\
&\quad \times (\varepsilon \bowtie S^{-3}(\underline{g}^{-1})) (\mu^{-1} \bowtie S^{-1}(g^1 x^3 \mathbb{G}^2)_2 S^{-1}(q^1 G^1)) \\
&\stackrel{(2.19), (2.8), (4.13)}{=} \mu(\tilde{q}^1 \mathbb{G}^1) \mu(S(\tilde{p}^1) f^2 \tilde{q}_2^2 \mathbb{G}_2^2 G^2) \\
&\quad \times (\varepsilon \bowtie S^{-3}(\underline{g}^{-1})) (\mu^{-1} \bowtie S^{-1}(S(\tilde{p}^2) f^1 \tilde{q}_1^2 \mathbb{G}_1^2 G^1)) \\
&\stackrel{(2.8)}{=} \mu(\tilde{q}^1 \mathbb{G}^1) \mu^{-1}(\tilde{p}^1) (\varepsilon \bowtie S^{-3}(\underline{g}^{-1})) (\mu^{-1} \bowtie (S^{-1}(\tilde{q}^2 \mathbb{G}^2) \leftarrow \mu^{-1}) \tilde{p}^2),
\end{aligned}$$

and this completes the proof. \square

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Appendix A

We will present the proofs of (5.12) and the fact that $T \bowtie r$ is a right integral in $D(H)$. Formula (5.12) can be viewed as the generalization to quasi-Hopf algebras of the well-known equality $\lambda(h_1)h_2 = \lambda(h)\underline{g}$, for $\lambda \in \mathcal{L}$ and $h \in H$, where H is a finite dimensional Hopf algebra. As we will see, it allows us to generalize Radford's result [17] that says that the Drinfeld double of a finite dimensional Hopf algebra is unimodular to quasi-Hopf algebras.

First we compute the inverse of the Nakayama isomorphism χ introduced in Proposition 4.1.

Lemma A.1. $\chi^{-1}(h) = \mu(S^{-1}(uhu^{-1})_2) S^{-1}(S^{-1}(uhu^{-1})_1)$, for all $h \in H$, where u is the element of H introduced in Proposition 4.3. Consequently,

$$\mu^{-1}(\tilde{q}^1 h_1 \tilde{p}^1) \lambda \leftarrow S^{-1}(\tilde{q}^2 h_2 \tilde{p}^2) = \mu^{-1}(\alpha) \mu(\beta) S(h) \rightharpoonup \lambda, \quad (\text{A.1})$$

for all $\lambda \in \mathcal{L}$ and $h \in H$.

Proof. It follows from Proposition 4.3 that

$$\begin{aligned}\lambda(\tilde{q}^2 t_2 \tilde{p}^2) \tilde{q}^1 t_1 \tilde{p}^1 &= \Lambda(uS(\tilde{q}^2 t_2 \tilde{p}^2)) \tilde{q}^1 t_1 \tilde{p}^1 \\ &\stackrel{(2.22)}{=} \mu(S^{-1}(u)_2) \Lambda(S(\tilde{q}^2 t_2 \tilde{p}^2)) \tilde{q}^1 t_1 \tilde{p}^1 S^{-1}(S^{-1}(u)_1).\end{aligned}$$

Hence, by (4.2) we obtain that

$$\begin{aligned}\chi^{-1}(h) &= \phi(hS(q^2 t_2 p^2)) q^1 t_1 p^1 = \lambda(q^2 t_2 p^2 S^{-1}(h)) q^1 t_1 p^1 \\ &= \lambda(\tilde{q}^2 t_2 \tilde{p}^2 S^{-1}(hu^{-1})) \tilde{q}^1 t_1 \tilde{p}^1 \\ &\stackrel{(2.22)}{=} \mu(S^{-1}(hu^{-1})_2) \lambda(\tilde{q}^2 t_2 \tilde{p}^2) \tilde{q}^1 t_1 \tilde{p}^1 S^{-1}(S^{-1}(hu^{-1})_1) \\ &= \mu(S^{-1}(uhu^{-1})_2) S^{-1}(S^{-1}(uau^{-1})_1),\end{aligned}$$

as needed. In particular, this implies that

$$\begin{aligned}S^{-1}\chi^{-1}S^2(h) &= \mu(S^{-1}(u^{-1})_2 S(h)_2 S^{-1}(u)_2) S^{-2}(S^{-1}(u^{-1})_1 S(h)_1 S^{-1}(u)_1) \\ &= \mu^{-1}(q_2^1 g^2 S(q^2)) \mu(V^1) \mu(q_{(1,2)}^1 g_2^1 S(V^2 h)_2) S^{-2}(q_{(1,1)}^1 g_1^1 S(V^2 h)_1) \\ &\stackrel{(2.8), (2.14), (2.1)}{=} \mu^{-1}(x^3 G^2 S(q^2 X^1)) \mu(V^1) \mu(x^2 G^1 S(V_1^2 h_1 X^2) f^2) \\ &\quad \times S^{-2}(x^1 q^1 S(V_2^2 h_2 X^3) f^1) \\ &\stackrel{(3.5), (2.8), (2.14)}{=} \mu^{-1}(x^3 G^2 S(q^2 X^1)) \mu(S^{-1}(F^2 Y^3 p^2) y^1) \mu(x^2 G^1 S(y^2 h_1 X^2) F^1 Y^2 p_2^1) \\ &\quad \times S^{-2}(x^1 q^1 S(y^3 h_2 X^3) Y^1 p_1^1) \\ &\stackrel{(2.17), (5.14)}{=} \mu(X^1 \beta) \mu(S^{-1}(F^2 p^2) y^1) \mu(S(y^2 h_1 X^2) F^1 p^1) S^{-1}(y^3 h_2 X^3) \\ &\stackrel{(2.18), (2.19)}{\stackrel{(2.15)}{=}} (\mu^{-1}(\alpha) \mu(\beta))^{-1} \mu^{-1}(\tilde{q}^1 h_1 \tilde{p}^1) S^{-1}(\tilde{q}^2 h_2 \tilde{p}^2),\end{aligned}$$

for all $h \in H$. Consequently,

$$\begin{aligned}S(h) \rightarrow \lambda &= S(h) \rightarrow \phi \circ S = (\phi \leftarrow S^2(h)) \circ S = (\chi^{-1} S^2(h) \rightarrow \phi) \circ S \\ &= (\chi^{-1} S^2(h) \rightarrow \lambda \circ S^{-1}) \circ S = \lambda \leftarrow S^{-1} \chi^{-1} S^2(h) \\ &= (\mu^{-1}(\alpha) \mu(\beta))^{-1} \mu^{-1}(\tilde{q}^1 h_1 \tilde{p}^1) \lambda \leftarrow S^{-1}(\tilde{q}^2 h_2 \tilde{p}^2),\end{aligned}$$

finishing our proof. \square

Proof of (5.12). Using the definitions of q_R , q_L , (2.3) and (2.5) we deduce easily that

$$X^1 \otimes S(X^2) \tilde{q}^1 X_1^3 \otimes \tilde{q}^2 X_2^3 = q^1 x_1^1 \otimes S(q^2 x_2^1) x^2 \otimes x^3. \quad (\text{A.2})$$

The Nakayama isomorphism ξ_{cop} for H^{cop} is given by

$$\xi_{\text{cop}}: H^* \rightarrow H, \quad \xi_{\text{cop}}(h^*) = h^*(S^{-1}(\tilde{q}^1 t_1 \tilde{p}^1)) \tilde{q}^2 t_2 \tilde{p}^2,$$

and is an isomorphism of left H -modules with inverse $\xi_{\text{cop}}^{-1}(h) = h \rightarrow \Lambda \circ S$, where Λ is a right coin-tegral on H satisfying $\Lambda(S(t)) = 1$, see Remarks 4.2 (i).

Now take $h \in H$ and $h^* = \xi_{\text{cop}}^{-1}(h)$. If $q^* := h^* \circ S^{-1}$ then

$$h = q^*(\tilde{q}^1 t_1 \tilde{p}^1) \tilde{q}^2 t_2 \tilde{p}^2 \stackrel{(4.7)}{=} q^*(q^1 t_1 p^1) q^2 t_2 p^2 S^{-1}(u),$$

where u is the element introduced in Proposition 4.3. Set $p_R = p^1 \otimes p^2 = P^1 \otimes P^2$ and $q_R = q^1 \otimes q^2 = Q^1 \otimes Q^2$ and compute

$$\begin{aligned} \Delta(hS^{-1}(u^{-1})) &= q^*(q^1 t_1 p^1) q_1^2 t_{(2,1)} p_1^2 \otimes q_2^2 t_{(2,2)} p_2^2 \\ &\stackrel{(2.29)}{=} q^*(q^1 Q_1^1 x^1 t_1 p^1) S^{-1}(g^2) q^2 Q_2^1 x^2 t_{(2,1)} p_1^2 \otimes S^{-1}(g^1) Q^2 x^3 t_{(2,2)} p_2^2 \\ &\stackrel{(2.28), (3.8)}{=} \mu(X^1) q^*(q^1 (Q^1 t_1 P^1)_1 p^1) S^{-1}(g^2) q^2 (Q^1 t_1 P^1)_2 p^2 S(X^3) f^1 \\ &\quad \otimes S^{-1}(g^1) Q^2 t_2 P^2 S(X^2) f^2. \end{aligned}$$

This equality is equivalent to

$$\begin{aligned} (S^{-1}(f^2) \otimes S^{-1}(f^1)) \Delta(hS^{-1}(u^{-1})) (g^1 \otimes g^2) \\ = \mu(X^1) q^*(q^1 (Q^1 t_1 P^1)_1 p^1) q^2 (Q^1 t_1 P^1)_2 p^2 S(X^3) \otimes Q^2 t_2 P^2 S(X^2). \end{aligned}$$

Applying $\lambda \otimes \text{Id}_H$ to this formula, we find

$$\begin{aligned} \lambda(S^{-1}(f^2)(hS^{-1}(u^{-1}))_1 g^1 S(h')) S^{-1}(f^1)(hS^{-1}(u^{-1}))_2 g^2 \\ = \mu(X^1) q^*(q^1 (Q^1 t_1 P^1)_1 p^1) \lambda(q^2 (Q^1 t_1 P^1)_2 p^2 S(h' X^3)) Q^2 t_2 P^2 S(X^2) \\ \stackrel{(3.17)}{=} \mu(x^1 X^1) q^*(\tilde{q}^2 x^3 h'_2 X_2^3 \tilde{p}^2) \lambda(S^{-1}(\tilde{q}^1) Q^1 t_1 P^1 S(x^2 h'_1 X_1^3 \tilde{p}^1)) Q^2 t_2 P^2 S(X^2). \end{aligned}$$

We know that $h^* = \xi_{\text{cop}}^{-1}(h) = h \rightharpoonup \Lambda \circ S$, hence $q^* = (h \rightharpoonup \Lambda \circ S) \circ S^{-1} = \Lambda \leftarrow S(h)$. By Proposition 4.3 we have that $\Lambda = (\lambda \circ S^{-1}) \leftarrow u^{-1}$, hence $q^* = (\lambda \circ S^{-1}) \leftarrow u^{-1} S(h)$, and this implies that

$$\begin{aligned} \lambda(S^{-1}(f^2)(hS^{-1}(u^{-1}))_1 g^1 S(h')) S^{-1}(f^1)(hS^{-1}(u^{-1}))_2 g^2 \\ = \mu(x^1 X^1) \lambda(S^{-1}(\tilde{q}^2 x^3 h'_2 X_2^3 \tilde{p}^2) h S^{-1}(u^{-1})) \lambda(S^{-1}(\tilde{q}^1) Q^1 t_1 P^1 S(x^2 h'_1 X_1^3 \tilde{p}^1)) Q^2 t_2 P^2 S(X^2). \end{aligned}$$

Since u is invertible it follows that

$$\begin{aligned} \lambda(S^{-1}(f^2) h_1 g^1 S(h')) S^{-1}(f^1) h_2 g^2 \\ = \mu(x^1 X^1) \lambda(S^{-1}(\tilde{q}^2 x^3 h'_2 X_2^3 \tilde{p}^2) h) \lambda(S^{-1}(\tilde{q}^1) Q^1 t_1 P^1 S(x^2 h'_1 X_1^3 \tilde{p}^1)) Q^2 t_2 P^2 S(X^2) \\ \stackrel{(3.19), (2.8)}{=} \mu(x^1 X^1 g^1 S(x_2^2 h'_{(1,2)} X_{(1,2)}^3 \tilde{p}_2^1) f^1) \lambda(S^{-1}(\tilde{q}^2 x^3 h'_2 X_2^3 \tilde{p}^2) h) \\ \quad \times \lambda(S^{-1}(\tilde{q}^1) Q^1 t_1 P^1) Q^2 t_2 P^2 S(X^2 g^2 S(x_1^2 h'_{(1,1)} X_{(1,1)}^3 \tilde{p}_1^1) f^2) \\ \stackrel{(2.30), (3.20)}{\stackrel{(2.1), (3.19)}{=}} \mu(\tilde{q}_1^1 x^1) \mu(S((\tilde{q}_2^1 x^2)_2 y^2 (h'_2 \tilde{p}^2)_1 \tilde{p}^1) f^1) \lambda(S^{-1}(\tilde{q}^2 x^3 y^3 (h'_2 \tilde{p}^2)_2 \tilde{p}^2) h) \\ \quad \times \lambda(Q^1 t_1 P^1) Q^2 t_2 P^2 S((\tilde{q}_2^1 x^2)_1 y^1 h'_1 \tilde{p}^1) f^2) \end{aligned}$$

$$\begin{aligned}
& \stackrel{(4.4), (2.31), (A.2)}{=} \lambda(S^{-1}(\tilde{q}^2 y^3 (x_2^3 h'_2 \tilde{p}^2)_2 \tilde{p}^2) h) \mu(S(G_2^2 S(q^1 x_1^1)_2 \tilde{q}_2^1 y^2 (x_2^3 h'_2 \tilde{p}^2)_1 \tilde{p}^1) f^1) \\
& \quad \times \mu(G^1 S(q^2 x_2^1) x^2) S^{-1}(\underline{g}^{-1}) S(S(G_1^2 S(q^1 x_1^1)_1 \tilde{q}_1^1 y^1 x_1^3 h'_1 \tilde{p}^1) f^2) \\
& \stackrel{(2.8), (2.31)}{=} \mu(G^1 S(q^2 x_2^1) x^2) \mu(S(G_2^2 g^2 S(X^1 q_1^1 x_{(1,1)}^1) \tilde{q}^1 (\tilde{Q}^2 X_2^3 x_2^3 h'_2 \tilde{p}^2)_1 \tilde{p}^1) f^1) \\
& \quad \times \lambda(S^{-1}(\tilde{q}^2 (\tilde{Q}^2 X_2^3 x_2^3 h'_2 \tilde{p}^2)_2 \tilde{p}^2) h) S^{-1}(\underline{g}^{-1}) \\
& \quad \times S(S(G_1^2 g^1 S(X^2 q_2^1 x_{(1,2)}^1) \tilde{Q}^1 X_1^3 x_1^3 h'_1 \tilde{p}^1) f^2) \\
& \stackrel{(A.2), (A.1), (2.21)}{=} \mu^{-1}(\alpha) \mu(\beta) \mu(G^1 S(q^2 (y_1^1 x^1)_2) y_2^1 x^2) \mu(S(G_2^2 g^2 S(Q^1 q_1^1 (y_1^1 x^1)_{(1,1)}) f^1) \\
& \quad \times \lambda(h S(y^3 x_2^3 h'_2 \tilde{p}^2)) S^{-1}(\underline{g}^{-1}) S(S(G_1^2 g^1 S(Q^2 q_2^1 (y_1^1 x^1)_{(1,2)}) y_2^1 x_1^3 h'_1 \tilde{p}^1) f^2) \\
& \stackrel{(2.29), (2.14)}{=} \mu^{-1}(\alpha) \mu(\beta) \mu(Y^1 g_1^1 G^1 S((q^2 y_2^1)_2) F^1 y_1^2 x^1) \mu(S(Y^3 g^2 S(q^1 y_1^1) f^1) \\
& \stackrel{(2.1), (2.3)}{=} \mu^{-1}(\alpha) \mu(\beta) \mu(Y^1 g_1^1 G^1 S((q^2 y_2^1)_2) F^1 y_1^2 x^1) \mu(S(Y^3 g^2 S(q^1 y_1^1) f^1) \\
& \quad \times \lambda(h S(y^3 x^3 h'_2 \tilde{p}^2)) S^{-1}(\underline{g}^{-1}) S(S(Y^2 g_2^1 G^2 S((q^2 y_2^1)_1) F^2 y_2^2 x^2 h'_1 \tilde{p}^1) f^2) \\
& \stackrel{(2.8), (2.1), (3.5)}{=} \mu^{-1}(\alpha) \mu(\beta) \mu(f^1) \mu(S(y^1)_1 Y^1 U_1^1 y_1^2 x^1) \mu^{-1}(S(y^1)_{(2,2)} Y^3 U^2) \\
& \quad \times \lambda(h S(y^3 x^3 h'_2 \tilde{p}^2)) S^{-1}(\underline{g}^{-1}) S(S(S(y^1)_{(2,1)} Y^2 U_2^1 y_2^2 x^2 h'_1 \tilde{p}^1) f^2) \\
& \stackrel{(4.5)}{=} \mu(\beta f^1) \mu^{-1}(Y^3 U^2 \alpha) \mu(Y^1 U_1^1 y_1^2 x^1) \lambda(h S(y^3 x^3 h'_2 \tilde{p}^2)) \\
& \quad \times S^{-1}(\underline{g}^{-1} y^1) S(S(Y^2 U_2^1 y_2^2 x^2 h'_1 \tilde{p}^1) f^2).
\end{aligned}$$

Thus we have shown that

$$\begin{aligned}
\lambda(S^{-1}(f^2) h_1 g^1 S(h')) S^{-1}(f^1) h_2 g^2 &= \mu(\beta f^1) \mu^{-1}(Y^3 U^2 \alpha) \mu(Y^1 U_1^1 y_1^2 x^1) \lambda(h S(y^3 x^3 h'_2 \tilde{p}^2)) \\
&\quad \times S^{-1}(\underline{g}^{-1} y^1) S(S(Y^2 U_2^1 y_2^2 x^2 h'_1 \tilde{p}^1) f^2), \tag{A.3}
\end{aligned}$$

for all $h, h' \in H$. Finally, by [11, Lemma 3.13] we have

$$x^1 U^1 \otimes x^2 U_1^1 \mathbf{U}^1 \otimes x^3 U_2^2 \mathbf{U}^2 = S(X^1)_{(1,1)} U_1^1 X^2 \otimes S(X^1)_{(1,2)} U_2^1 X^3 \otimes S(X^1)_2 U^2.$$

Substituting this formula in (A.3) and applying (4.5), (2.3), we easily obtain (5.12), as desired. \square

Our proof for the fact that $T \bowtie r$ is a right integral in $D(H)$ requires the following formulas.

Lemma A.2. Let H be a finite dimensional quasi-Hopf algebra, $h \in H$ and $r \in \int_r^H$. Then

$$\begin{aligned}
& X_1^1 x^1 \delta^1 S(X_2^3) \otimes X_2^1 x^2 \delta_1^2 S(X_1^3)_1 \otimes X^2 x^3 \delta_2^2 S(X_1^3)_2 \\
& = (\beta S(X^3))_1 g^1 S(x^3) \otimes (\beta S(X^3))_2 g^2 S(x^2) f^1 \otimes x^2 X^1 \beta S(x^1 X^2) f^2, \tag{A.4}
\end{aligned}$$

$$f^2 V^1 S^{-1}(f^1)_1 \otimes V^2 S^{-1}(f^1)_2 = q_L, \tag{A.5}$$

$$S(U^1) \tilde{q}^1 U_1^2 \otimes \tilde{q}^2 U_2^2 = f, \tag{A.6}$$

$$S(p^1) F^2 f_2^2 X^3 \otimes S(p^2 f^1 X^1) F^1 f_1^1 X^2 = 1 \otimes \alpha, \tag{A.7}$$

$$V^1 r_1 \otimes \underline{g}^{-1} V^2 r_2 = V^2 r_2 p^2 \otimes S^2(V^1 r_1 p^1) \alpha, \tag{A.8}$$

$$S(\tilde{p}^2) f^1 r_1 \otimes \underline{g}^{-1} S(\tilde{p}^1) f^2 r_2 = \mu(S(p^2) f^1) S(p^1) f^2 V^2 r_2 p^2 \otimes S^2(V^1 r_1 p^1) \alpha. \tag{A.9}$$

Proof. We have

$$\begin{aligned}
 & X_1^1 x^1 \delta^1 S(X_2^3) \otimes X_2^1 x^2 \delta_1^2 S(X_1^3)_1 \otimes X^2 x^3 \delta_2^2 S(X_1^3)_2 \\
 & \stackrel{(2.13), (2.8), (2.1)}{=} (X^1 \beta_1)_1 x^1 g^1 S(X_2^3) \otimes (X^1 \beta_1)_2 x^2 g_1^2 G^1 S(X_{(1,2)}^3) f^1 \otimes X^2 \beta_2 x^3 g_2^2 G^2 S(X_{(1,1)}^3) f^2 \\
 & \stackrel{(2.14), (2.1)}{=} (X^1 \beta_1 g^1)_1 G^1 S(x^3 X_{(2,2)}^3) \otimes (X^1 \beta_1 g^1)_2 G^2 S(x^2 X_{(2,1)}^3) f^1 \otimes X^2 \beta_2 g^2 S(x^2 X_1^3) f^2 \\
 & \stackrel{(2.13), (2.8)}{=} (X^1 \delta^1 S(X_2^3))_1 G^1 S(x^3) \otimes (X^1 \delta^1 S(X_2^3))_2 G^2 S(x^2) f^1 \otimes X^2 \delta^2 S(x^1 X_1^3) f^2 \\
 & \stackrel{(2.10)}{=} (\beta S(X^3))_1 g^1 S(x^3) \otimes (\beta S(X^3))_2 g^2 S(x^2) f^1 \otimes x^2 X^1 \beta S(x^1 X^2) f^2,
 \end{aligned}$$

and this proves (A.4). The equalities (A.5) and (A.7) follow from the definitions of V and p_R , and from (2.14), (2.15). The verification of all of these details is left to the reader. We show now (A.6) by computing

$$\begin{aligned}
 & S(U^1) \tilde{q}^1 U_1^2 \otimes \tilde{q}^2 U_2^2 \stackrel{(4.11)}{=} S(\tilde{Q}_1^1 p^1) \tilde{q}^1 \tilde{Q}_{(2,1)}^1 p_1^2 S(\tilde{Q}^2)_1 \otimes \tilde{q}^2 \tilde{Q}_{(2,2)}^1 p_2^2 S(\tilde{Q}^2)_2 \\
 & \stackrel{(2.23)}{=} S(p^1) \tilde{q}^1 p_1^2 S(\tilde{Q}^2)_1 \otimes \tilde{Q}^1 \tilde{q}^2 p_2^2 S(\tilde{Q}^2)_2 \\
 & = S(\tilde{p}^2) f^1 S(\tilde{Q}^2)_1 \otimes \tilde{Q}^1 S(\tilde{p}^1) f^1 S(\tilde{Q}^2)_2 \\
 & \stackrel{(2.8)}{=} S(\tilde{Q}_2^2 \tilde{p}^2) f^1 \otimes \tilde{Q}^1 S(\tilde{Q}_1^2 \tilde{p}^1) f^2 \stackrel{(2.27)}{=} f.
 \end{aligned}$$

In the third equality we have used the second equation in (4.11), applied to H^{cop} , namely $\tilde{q}^2 p_2^2 \otimes S(p^1) \tilde{q}^1 p_1^2 = V_{\text{cop}} = S(\tilde{p}^1) f^2 \otimes S(\tilde{p}^2) f^1$.

In order to show (A.8), notice that (4.10) and (4.3) imply that

$$V^1 r_1 U^1 \otimes \underline{g}^{-1} V^2 r_2 U^2 = V^2 r_2 U^2 \otimes S^2(V^1 r_1 U^1).$$

As we have already observed, (4.11) guarantees that $\Delta(r)U = \Delta(r)p_R$, and so

$$\begin{aligned}
 & V^1 r_1 \otimes \underline{g}^{-1} V^2 r_2 \stackrel{(2.25)}{=} V^1 r_1 q_1^1 p^1 \otimes \underline{g}^{-1} V^2 r_2 q_2^1 p^2 S(q^2) \\
 & = V^1 r_1 p^1 \otimes \underline{g}^{-1} V^2 r_2 p^2 \alpha = V^2 r_2 p^2 \otimes S^2(V^1 r_1 p^1) \alpha,
 \end{aligned}$$

as required. Finally, we have

$$\begin{aligned}
 & S(\tilde{p}^2) f^1 r_1 \otimes \underline{g}^{-1} S(\tilde{p}^1) f^2 r_2 \stackrel{(4.13)}{=} \mu^{-1}(g^1) q^1 r_1 \otimes S^{-1}(g^2) q^2 r_2 \\
 & \stackrel{(3.14)}{=} \mu^{-1}(g^1) \mu(\tilde{q}^1) \tilde{q}^2 V^1 r_1 \otimes S^{-1}(g^2) V^2 r_2 \\
 & \stackrel{(5.18)}{=} \mu(S(g^1) \tilde{q}^1 g_1^2) \tilde{q}^2 g_2^2 V^1 r_1 \otimes \underline{g}^{-1} V^2 r_2 \\
 & \stackrel{(A.8), (4.14)}{=} \mu(S(p^2) f^1) S(p^1) f^2 V^2 r_2 p^2 \otimes S^2(V^1 r_1 p^1) \alpha,
 \end{aligned}$$

proving (A.9). This makes the proof complete. \square

We are now ready to prove our final result, stating that the Drinfeld double of a finite dimensional quasi-Hopf algebra is unimodular.

Proposition A.3. *If H is a finite dimensional quasi-Hopf algebra, $0 \neq r \in \int_r^H$ and $0 \neq \lambda \in \mathcal{L}$ then $\mathbb{T} = \mu^{-1}(\delta^2)\delta^1 \rightharpoonup \lambda \bowtie r$ is a non-zero right integral in $D(H)$. Consequently, $D(H)$ is a unimodular quasi-Hopf algebra.*

Proof. By the definitions of p_L and p_R , and the axioms (2.3), (2.5) we obtain that

$$x^1 \otimes x^2 S(x_1^3 \tilde{p}^1) \otimes x_2^3 \tilde{p}^2 = X_1^1 p^1 \otimes X_2^1 p^2 S(X^2) \otimes X^3. \quad (\text{A.10})$$

Now we can prove directly that \mathbb{T} is a right integral in $D(H)$: for $\varphi \in H^*$ and $h \in H$, we compute that

$$\begin{aligned} \mathbb{T}(\varphi \bowtie h) &\stackrel{(5.2), (5.1)}{=} \stackrel{(3.20), (3.9)}{=} (y^1 X_1^1 x^1 \delta^1 S(X_2^3) \rightharpoonup \lambda \leftarrow S^{-1}(f^2))(y^2 (X_2^1 x^2 \delta_1^2 S(X_1^3))_1) \\ &\quad \times r_{(1,1)} \rightharpoonup \varphi \leftarrow S^{-1}(f^1 X^2 x^3 \delta_2^2 S(X_1^3)_2) \bowtie y^3 (X_2^1 x^2 \delta_1^2 S(X_1^3))_2 r_{(1,2)} h \\ &\stackrel{(A.4), (2.1)}{=} \stackrel{(2.8), (2.14)}{=} ((\beta S(X^3))_{(1,1)} g_1^1 G^1 S(x^3 y^3) \rightharpoonup \lambda \leftarrow S^{-1}(\mathbb{F}^2)) ((\beta S(X^3))_{(1,2)} \\ &\quad \times g_2^1 G^2 S(x_2^2 y^2) F^1 f_1^1 r_{(1,1)} \rightharpoonup \varphi \leftarrow S^{-1}(\mathbb{F}^1 X^1 \beta S(x^1 X^2) f^2 r_2)) \\ &\quad \bowtie (\beta S(X^3))_2 g^2 S(x_1^2 y^1) F^2 f_2^1 r_{(1,2)} h \\ &\stackrel{(2.8), (2.13)}{=} ((\delta^1 S(X_2^3))_1 G^1 S(x^3 y^3) \rightharpoonup \lambda \leftarrow S^{-1}(\mathbb{F}^2)) ((\delta^1 S(X_2^3))_2 G^2 \\ &\quad \times S(x_2^2 y^2) F^1 f_1^1 r_{(1,1)} \rightharpoonup \varphi \leftarrow S^{-1}(\mathbb{F}^1 X^1 \beta S(x^1 X^2) f^2 r_2)) \\ &\quad \bowtie \delta^2 S(x_1^2 y^1 X_1^3) F^2 f_2^1 r_{(1,2)} h \\ &\stackrel{(5.12)}{=} \varphi(S^{-1}(\underline{g}^{-1} z_1^1 t^1 X^1 \beta S(x^1 X^2) f^2 r_2) S(x_2^2 y^2 S(U_1^2 \mathbf{U}^1 z^2 t_1^3 x_1^3 y_1^3 \tilde{p}^1) \mathbb{F}^2) \\ &\quad \times F^1 f_1^1 r_{(1,1)}) \mu(\beta \mathbb{F}^1) \mu^{-1}(U_2^2 \mathbf{U}^2 \alpha) \mu(U^1 z_2^1 t^2) \delta^1 S(z^3 t_2^3 x_2^3 y_2^3 \tilde{p}^2 X_2^3) \rightharpoonup \lambda \\ &\quad \bowtie \delta^2 S(x_1^2 y^1 X_1^3) F^2 f_2^1 r_{(1,2)} h \\ &\stackrel{(2.3), (2.22)}{=} \varphi(S^{-1}(\underline{g}^{-1} z_1^1 t^1 X^1 \beta S(Y^1 y_1^1 x^1 X^2) f^2 r_2) S(Y^3 y^2 S(U_1^2 \mathbf{U}^1 z^2 t_1^3 y_1^3 \tilde{p}^1) \mathbb{F}^2) \\ &\quad \times F^1 f_1^1 r_{(1,1)}) \mu(\beta \mathbb{F}^1) \mu^{-1}(U_2^2 \mathbf{U}^2 \alpha) \mu(U^1 z_2^1 t^2) \delta^1 S(z^3 t_2^3 y_2^3 \tilde{p}^2 x^3 X_2^3) \rightharpoonup \lambda \\ &\quad \bowtie \delta^2 S(Y^2 y_2^1 x^2 X_1^3) F^2 f_2^1 r_{(1,2)} h \\ &\stackrel{(A.10), (2.1)}{=} \stackrel{(3.9), (2.3)}{=} \varphi(S^{-1}(\underline{g}^{-1} z_1^1 t^1 X^1 \beta S(Y^1 p_1^1 x^1 X^2) f^2 r_2) S(Y^3 p^2 S(U_1^2 \mathbf{U}^1 z^3 t_2^2) \mathbb{F}^2) \\ &\quad \times F^1 f_1^1 r_{(1,1)}) \mu(\beta \mathbb{F}^1) \mu^{-1}(U_2^2 \mathbf{U}^2 \alpha) \mu(U^1 z^2 t_1^2) \delta^1 S(t^3 x^3 X_2^3) \rightharpoonup \lambda \\ &\quad \bowtie \delta^2 S(Y^2 p_2^1 x^2 X_1^3) F^2 f_2^1 r_{(1,2)} h \\ &\stackrel{(3.9), (2.8), (2.1)}{=} \varphi(S^{-1}(\underline{g}^{-1} z_1^1 t^1 X^1 \beta S(Y^1 (z_1^2 t_{(1,1)}^2 p^1)_1 x^1 X^2) f^2 r_2) \\ &\quad \times S(Y^3 z_2^2 t_{(1,2)}^2 p^2 S(U_1^2 \mathbf{U}^1 z^3 t_2^2) \mathbb{F}^2) F^1 f_1^1 r_{(1,1)}) \mu(\beta \mathbb{F}^1) \mu^{-1}(U_2^2 \mathbf{U}^2 \alpha) \\ &\quad \times \mu(U^1) \delta^1 S(t^3 x^3 X_2^3) \rightharpoonup \lambda \bowtie \delta^2 S(Y^2 (z_1^2 t_{(1,1)}^2 p^1)_2 x^2 X_1^3) F^2 f_2^1 r_{(1,2)} h \\ &\stackrel{(2.20), (2.3)}{=} \varphi(S^{-1}(\underline{g}^{-1} z_1^1 X^1 \beta S(Y^1 (z_1^2 p^1)_1 X^2) f^2 r_2) S(Y^3 z_2^2 p^2 S(U_1^2 \mathbf{U}^1 z^3) \mathbb{F}^2) \\ &\quad \times F^1 f_1^1 r_{(1,1)}) \mu(\beta \mathbb{F}^1) \mu^{-1}(U_2^2 \mathbf{U}^2 \alpha) \mu(U^1) \delta^1 \rightharpoonup \lambda \\ &\quad \bowtie \delta^2 S(Y^2 (z_1^2 p^1)_2 X^3) F^2 f_2^1 r_{(1,2)} h \end{aligned}$$

$$\begin{aligned}
& \stackrel{(2.1), (5.14), (2.20)}{=} \varphi(S^{-1}(\underline{g}^{-1}z^1X^1\beta S(z_1^2t^1X^2)f^2r_2)S((z_2^2t^2X_1^3)_2p^2S(U_1^2\mathbf{U}^1z^3t^3X_2^3)\mathbb{F}^2) \\
& \quad \times F^1f_1^1r_{(1,1)})\mu(\beta\mathbb{F}^1)\mu^{-1}(U_2^2\mathbf{U}^2\alpha)\mu(U^1)\delta^1 \rightarrow \lambda \\
& \quad \bowtie \delta^2 S((z_2^2t^2X_1^3)_1p^1)F^2f_2^1r_{(1,2)}h \\
& \stackrel{(2.3), (2.18), (2.8)}{=} \varphi(S^{-1}(\underline{g}^{-1}S(\tilde{p}^1)f^2r_2)S(p^2S(U_1^2\mathbf{U}^1)\mathbb{F}^2)F^1(S(\tilde{p}^2)f^1r_1)_1)\mu(\beta\mathbb{F}^1) \\
& \quad \times \mu^{-1}(U_2^2\mathbf{U}^2\alpha)\mu(U^1)\delta^1 \rightarrow \lambda \bowtie \delta^2 S(p^1)F^2(S(\tilde{p}^2)f^1r_1)_2h \\
& \stackrel{(A.9), (2.8)}{=} \varphi(S^{-1}(\alpha)S(p^2S(\mathbf{U}^1)\mathbb{F}^2S(U^2)_2V^1r_1\mathbb{P}^1)F^1(S(P^1)f^2V^2r_2\mathbb{P}^2)_1) \\
& \quad \times \mu(S(P^2)f^1)\mu(\beta\mathbb{F}^1)\mu^{-1}(U^2\alpha)\mu(S(U^2)_1)\delta^1 \rightarrow \lambda \\
& \quad \bowtie \delta^2 S(p^1)F^2(S(P^1)f^2V^2r_2\mathbb{P}^2)_2h \\
& \stackrel{(3.9), (5.18)}{=} \mu(\beta\mathbb{F}^1)\mu^{-1}(\beta)\mu(\alpha)\varphi(S^{-1}(\alpha)S(p^2S(\mathbf{U}^1)\mathbb{F}^2V^1r_1\mathbb{P}^1)F^1(V^2r_2\mathbb{P}^2)_1) \\
& \stackrel{(3.5), (2.17)}{=} \mu^{-1}(\mathbf{U}^2\alpha)\delta^1 \rightarrow \lambda \bowtie \delta^2 S(p^1)F^2(V^2r_2\mathbb{P}^2)_2h \\
& \stackrel{(3.9), (A.5), (3.10)}{=} \mu^{-1}(\mathbf{U}^2)\varphi(S^{-1}(\alpha)S(p^2S(\mathbf{U}^1)\tilde{q}^1r_1\mathbb{P}^1)F^1(\tilde{q}^2r_2\mathbb{P}^2)_2) \\
& \quad \times \delta^1 \rightarrow \lambda \bowtie \delta^2 S(p^1)F^2(\tilde{q}^2r_2\mathbb{P}^2)_2h \\
& \stackrel{(3.9), (A.6)}{=} \varphi(S^{-1}(\alpha)S(p^2f^1X^1(r_1p^1)_1P^1)F^1f_1^2X^2(r_1p^1)_2P^2) \\
& \stackrel{(2.28), (2.1)}{=} \varphi(S^{-1}(\alpha)S(p^2f^1X^1(r_1p^1)_1P^1)F^1f_1^2X^2(r_1p^1)_2P^2) \\
& \quad \times \delta^1 \rightarrow \lambda \bowtie \delta^2 S(p^1)F^2f_2^2X^3r_2p^2h \\
& \stackrel{(A.7)}{=} \varphi(S^{-1}(\alpha)S((r_1p^1)_1P^1)\alpha(r_1p^1)_2P^2)\delta^1 \rightarrow \lambda \bowtie \delta^2 r_2p^2h \\
& \stackrel{(2.5), (2.6)}{=} \varphi(S^{-1}(\alpha))\delta^1 \rightarrow \lambda \bowtie \delta^2 r\beta h \\
& = \varphi(S^{-1}(\alpha))\varepsilon(h)\mu^{-1}(\delta^2) \rightarrow \lambda \bowtie r = \varepsilon_D(\varphi \bowtie h)T \bowtie r,
\end{aligned}$$

as required. This finishes the proof. \square

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